GENERALIZED CURVES

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The present paper is the first of a sequence of three, the principal object of the sequence being the establishment of existence theorems for Bolza problems in the calculus of variations. These existence theorems will be of sufficient generality to apply to problems with fixed end points and with variable end points, and also to include isoperimetric problems.

The scheme of proof here adopted might be called the method of the auxiliary problem. In very broad outline, it is as follows. The problem is to minimize a functional $f(x)$ on a class S of elements x. To show the existence of a solution, the range of the functional f is extended to a larger class S^* of elements possessing more desirable properties than S; specifically, a kind of local compactness. By use of these properties of S^* it is shown that there exists an element x_0 of S^* which minimizes $f(x)$ on S^*. Next we find the necessary conditions which are satisfied by x_0 as a consequence of minimizing $f(x)$. Under suitable hypotheses on f these conditions will imply that x_0 is a member, not merely of the extended class S^*, but of the original class S. Thus x_0 is the minimizing element which is the solution of the original problem.

In this note and its two successors we shall study single-integral problems by introducing an auxiliary problem. The original class S of curves will be enlarged to the class S^* of generalized curves, invented by L. C. Young;[1,2,3] and each of the three papers of the sequence will develop one of the three steps outlined in the general proof-pattern. In this first paper we develop the theory of generalized curves, and show that they possess a compactness property which leads us readily to an existence theorem in the class of generalized curves. In the second paper we shall develop the theory of the calculus of variations as extended to generalized curves, obtaining (for Bolza problems) analogues of the multiplier rule, the Weierstrass and Clebsch conditions, and the Dresden corner condition. In the third paper we set forth additional hypotheses on the integrands involved which guarantee that the minimizing generalized curve is actually a curve in the ordinary sense. In all three papers we shall consider only problems in parametric form.

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[1] L. C. Young, *On approximation by polygons in the calculus of variations*, Proceedings of the Royal Society, (A), vol. 141(1933), pp. 325–341.

[2] L. C. Young, *Generalized curves and the existence of an attained absolute minimum in the calculus of variations*, Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, Classe III, vol. 30(1937), pp. 212–234.

[3] L. C. Young, *Necessary conditions in the calculus of variations*, Acta Math., vol. 69(1938), pp. 239–258.