CYCLIC TRANSITIVITY

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Introduction and fundamental concepts

0.1. Let us denote by 1 a set which will serve as our space; the elements of the set 1 will be called points. However, we shall not assume that the set 1 is topologized in any way; that is, 1 is a wholly unconditioned set, unless a statement to the contrary is explicitly made.

0.2. Given in 1 a binary relation $\mathfrak{R}$, we shall write $a \mathfrak{R} b$ to express the fact that the points a and b of 1 are in the $\mathfrak{R}$-relation. Many important binary relations arising in algebra are reflexive, symmetric, and transitive; that is, $a \mathfrak{R} a$ for every point a; $a \mathfrak{R} b$ implies $b \mathfrak{R} a$ for every pair of points a, b; and $a \mathfrak{R} b \mathfrak{R} c$ implies $a \mathfrak{R} c$ for every triple of points a, b, c. On the other hand, the general theory of sets leads to binary relations—such as set inclusion—which are transitive, but are neither reflexive nor symmetric.[1] Binary relations of the types just mentioned have been studied and applied extensively. Both of these types are transitive. In this paper we are concerned with binary relations which are reflexive and symmetric, but are not necessarily transitive; the requirement of transitivity is replaced by a weaker condition which we shall call *cyclic transitivity*, and which we now describe.

0.3. Given a binary relation $\mathfrak{R}$ in 1, we say that $\mathfrak{R}$ is *cyclically transitive* if, for every finite cyclically ordered set of distinct points $a_1, a_2, \cdots, a_n$ satisfying $a_1 \mathfrak{R} a_2 \mathfrak{R} \cdots \mathfrak{R} a_n \mathfrak{R} a_1$, we have $a_i \mathfrak{R} a_j$ for every choice of the subscripts i and j. Let $\mathfrak{R}$ be a reflexive and symmetric binary relation; if $\mathfrak{R}$ is transitive (cf. 0.2), then clearly $\mathfrak{R}$ is cyclically transitive, but the converse is not true. Thus cyclic transitivity is an extension of ordinary transitivity, that is, an extension of one of the fundamental concepts arising in algebra. On the other hand, we shall see presently (cf. 0.4) that cyclic transitivity also arises in connection with certain fundamental concepts in topology.

Received February 26, 1940; presented to the American Mathematical Society, December 26, 1939. This is a condensed version of our original paper which was accepted for publication by the Fundamenta Mathematicae in August, 1939. In this paper we tried to arrange the definitions, lemmas, and theorems in such an order that the reader may construct the proofs for himself with the aid of hints given. Explicit proofs are given only in a few cases where the proof depends upon a device which might not readily occur to the reader.

[1] For an extensive and detailed study of transitive relations, see, for example, Foradori [1], [2], [3]. (Numbers in square brackets indicate references in the bibliography at the end of this paper.) We want to thank Professor Rainich for these references.