ANNIHILATORS OF QUADRATIC FORMS WITH APPLICATIONS TO PFAFFIAN SYSTEMS

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Introduction. This paper develops an algebraic approach to the study of certain arithmetic invariants of Pfaffian systems, thereby furnishing an extension of results previously obtained in connection with these invariants.[1] The principal algebraic result (Theorem 3.1) states that two quadratic forms defining a pencil of half-rank ρ in a Grassmann ring are simultaneously annihilated by the product of ρ linear forms. This result is employed to construct Pfaffian systems with half-rank ρ and species σ for all positive integers ρ, σ satisfying $\rho \leqq \sigma \leqq 2\rho$. This disproves a conjecture of Dearborn.[2] Finally we give a new upper bound for the species σ of a Pfaffian system of r equations, namely, $\sigma \leqq 2\rho + r - 1$.

1. **Pencils of forms.** By adjoining non-commutative marks $u_1, u_2, \cdots, u_n$ to a commutative field $\mathfrak{R}$ we obtain a Grassmann ring[3] which will be denoted by $\mathfrak{G}$.

Let S be a set of non-zero forms in $\mathfrak{G}$. S will be called a *pencil* if $a\omega + b\phi$ belongs to S whenever all the following three conditions are satisfied:

(i) a, b belong to $\mathfrak{R}$;

(ii) ω, ϕ belong to S;

(iii) $a\omega + b\phi \neq 0$.

The following properties of a pencil S follow directly from the definition of a pencil or are easily proved:

(a) Every member of S has non-negative degree.

(b) All members of S have the same degree.

(c) If S is a pencil, there is a positive integer r such that all members of S are given by

$$a_1\omega_1 + a_2\omega_2 + \cdots + a_r\omega_r ,$$

where the a's range over $\mathfrak{R}$ independently, but are not simultaneously zero.

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[1] See, for example, J. M. Thomas, *Pfaffian systems of species one*, Trans. Amer. Math. Soc., vol. 35(1933), pp. 356–371; Mabel Griffin, *Invariants of Pfaffian systems*, Trans. Amer. Math. Soc., vol. 35(1933), pp. 929–939; Donald Dearborn, *Inequalities among the invariants of Pfaffian systems*, this Journal, vol. 2(1936), pp. 705–711; J. M. Thomas, *A lower limit for the species of a Pfaffian system*, Proc. Nat. Acad. Sci., vol. 19(1933), p. 913.

[2] Loc. cit., p. 711.

[3] For a discussion of Grassmann algebra see J. M. Thomas, *Differential Systems*, New York, 1937, p. 10.