A CLASS OF INEQUALITIES

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Chapter I

F. Carlson[1] has proved the following inequality:

$$\left(\sum_1^\infty a_n\right)^4 < \pi^2 \sum_1^\infty a_n^2 \cdot \sum_1^\infty n^2 a_n^2, \qquad a_n \geqq 0. \tag{1.1}$$

He has also shown that π^2 is the best constant. Finally, he has shown that the inequality (1.1) may be considered as a limiting case of a Hölder inequality. This last statement is of particular importance to us and we shall return to it shortly. It is known that Carlson discovered (1.1) while engaged in studies in the theory of analytic functions. We can show that (1.1) has interesting consequences in this theory. For instance, consider the function

$$f(z) = \sum_1^\infty a_n z^n$$

with radius of convergence R. Then, for $r < R$,

$$\{M(r)\}^2 \leqq \pi r \mathfrak{M}_2(f) \mathfrak{M}_2(f'),$$

where $M(r)$ is the maximum value of $|f|$ on $|z| = r$ and $\mathfrak{M}_2(f)$, $\mathfrak{M}_2(f')$ represent the quadratic means of f and f' respectively on the circle. To obtain this result we need only take $c_n = |a_n| r^n$ and use (1.1).

We shall now show how the Carlson result may be considered as a limiting case of a Hölder inequality. The Hölder inequality gives

$$\sum_1^\infty a_n \leqq \left(\sum_1^\infty a_n^2\right)^{\frac{1}{4}} \left(\sum_1^\infty n^{2h} a_n^2\right)^{\frac{1}{4}} \left(\sum_1^\infty n^{-h}\right)^{\frac{1}{2}}.$$

Putting

$$K(h) = \left(\sum_1^\infty n^{-h}\right)^{\frac{1}{2}},$$

we can write

$$\sum_1^\infty a_n \leqq K(h) \left(\sum_1^\infty a_n^2\right)^{\frac{1}{4}} \left(\sum_1^\infty n^{2h} a_n^2\right)^{\frac{1}{4}},$$

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[1] F. Carlson, *Une inégalité*, Arkiv för Matematik, Astronomi och Fysik, vol. 25B(1934), no. 1, pp. 1–3.