THE DIFFERENCE OF CONSECUTIVE PRIMES

By P. Erdös

Let p_n denote the n-th prime. Backlund [1][1] proved that, for every positive ϵ and infinitely many n, $p_{n+1} - p_n > (2 - \epsilon) \log p_n$. Brauer and Zeitz [2, 10] proved that $2 - \epsilon$ can be replaced by $4 - \epsilon$. Westzynthius [9] proved that for an infinity of n

$$p_{n+1} - p_n > \frac{2 \log p_n \log \log \log p_n}{\log \log \log \log p_n},$$

and this was improved by Ricci [7] to

$$p_{n+1} - p_n > c_1 \log p_n \log \log \log p_n,$$

where, as throughout the paper, the c's denote positive absolute constants. I [4] showed that

$$p_{n+1} - p_n > c_2 \frac{\log p_n \log \log p_n}{(\log \log \log p_n)^2},$$

and lately Rankin [6] proved

$$p_{n+1} - p_n > c_3 \frac{\log p_n \log \log p_n \log \log \log \log p_n}{(\log \log \log p_n)^2}.$$

In the other direction the best known result is that of Ingham [5] which states that for sufficiently large n

$$p_{n+1} - p_n < p_n^{\frac{48}{77}+\epsilon} < p_n^{\frac{5}{8}}.$$

Thus it is known that

$$\limsup_{n \to \infty} \frac{p_{n+1} - p_n}{\log p_n} = \infty.$$

Very much less is known about

$$A = \liminf \frac{p_{n+1} - p_n}{\log p_n}.$$

Hardy and Littlewood proved a few years ago, by using the Riemann hypothesis, that $A \leqq \frac{2}{3}$, and Rankin recently proved, again by using the Riemann hypothesis, that $A \leqq \frac{3}{5}$. In the present paper we are going to prove—without the Riemann hypothesis—that

$$A < 1 - c_4, \qquad \text{for a certain } c_4 > 0.$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.