ON A THEOREM OF P. A. SMITH CONCERNING FIXED POINTS FOR PERIODIC TRANSFORMATIONS

BY SAMUEL EILENBERG

1. **Introduction.** The object of this paper is the discussion and generalization of the following theorem due to Smith:[1]

Let X be a point set in a Cartesian R^m, and Λ a topological transformation of X into itself of a finite and prime[2] period p. If every continuous single-valued image in X of every sphere of dimension $\leqq pm - m - 1$ is deformable in X to a point, then Λ leaves fixed at least one point.

The homotopy condition in this theorem is going to be replaced by a homology condition, expressed in terms of true cycles[3] in X with coefficients from a commutative ring R with a unit element.

Given $x \epsilon R$ and an integer n, we shall say that x is an *inverse* of n if $nx = 1$ (1 being the unit element of R).

THEOREM I. *Let X be a metric separable space of finite dimension, and Λ a topological transformation of X into itself of a finite and prime period p. Let R be a commutative ring with a unit element, which does not contain an inverse of p. If every true cycle in X with coefficients in R bounds in X, then Λ leaves fixed at least one point.*

If we take R to be the ring of all integers reduced mod q ($q = 0, 2, 3, \cdots$), it is easy to verify that the prime p has no inverse in R if and only if q is a multiple of p. We therefore obtain

THEOREM Ia. *Let X be a metric separable space of finite dimension, and Λ a topological transformation of X into itself of a finite and prime period p. Let q be any multiple of p (including 0). If every true cycle in X with coefficients* mod q *bounds in X, then Λ leaves fixed at least one point.*

True chains and true cycles can be replaced by singular chains and singular cycles throughout. Moreover, if X is a subset of a Cartesian R^m, then only the

Received December 7, 1939.

[1] P. A. Smith, *A theorem on fixed points for periodic transformations*, Annals of Math., vol. 35(1934), pp. 572–578.

[2] It is clear from Smith's proof that p must be a prime though this is not stated in his theorem.

[3] A sequence $C^n = \{c_1^n, c_2^n, \cdots\}$ is called an n-dimensional *true chain* in X if there exist a compact subset Y of X and a sequence of numbers $\epsilon_i \to 0$ such that c_i^n is an n-dimensional ϵ_i-chain in Y. C^n is a *true cycle* if $\partial C^n = 0$, where $\partial C^n = \{\partial c_1^n, \partial c_2^n, \cdots\}$ and ∂ is the usual boundary operator. C^n *bounds* if $C^n = \partial C^{n+1}$ for some true chain C^{n+1} in X. It is convenient in this paper to accept the convention that a true 0-chain $C^0 = \{c_1^0, c_2^0, \cdots\}$ is a 0-cycle only if the sum of coefficients in c_i^0 is 0 for $i = 1, 2, \cdots$.