GILLESPIE MEASURE

By Anthony P. Morse and John F. Randolph

1. **Remarks.** One aim of the theory of measure as introduced by Lebesgue was to furnish a tool for handling questions of length for curves and area for surfaces. While this theory was successful as far as curve length is concerned, the notion of area for surfaces was left in an unsatisfactory state. The fact that some of the properties of lengths for curves have not been successfully extended to areas for surfaces may suggest that our present notion of length for curves is perhaps of too special a nature. In fact even before Lebesgue had given his definition of the measure of a point set on a straight line, Minkowski [9][1] in 1909 had already considered the question of generalizing curve length by assigning (in the spirit of Peano-Jordan content) a linear measure to point sets lying in the plane.

This notion of assigning a linear measure to point sets not lying on a line has been considered in different ways since Minkowski with varying degrees of success. Among such definitions are those of Young 1905 [13], Janzen 1907 [7], Carathéodory 1914 [3], and Gross 1918 [5]. The Minkowski measure inherits all the faults of Peano-Jordan content. Young himself found inconsistencies for his own measure. As shown by Saks [11] Gross measure has the anomaly of assigning the measure zero to a particular set and a positive (even infinite) measure to the transform of this set by a bounded transformation of the plane. Janzen measure is not independent of the coördinate system, e.g., the set constructed by Gross ([5], p. 185) has Janzen measure unity for the axes used in the construction and measure zero when the axes are rotated 45°.

Until now Carathéodory measure has received practically no adverse criticism in the literature [8, 10] and seems to have been accepted as an adequate generalization of the notion of length. However, we give incidentally (§6) a set for which even this measure is quite inconsistent with our inherent concept of length.

2. **Introduction.** In this paper we propose still another definition of linear measure for point sets not necessarily lying on a line. For a point set A we shall represent this measure by $G^*(A)$ and call it Gillespie linear measure after the late Professor D. C. Gillespie who suggested to us individually definitions similar to the one we have adopted.

Gillespie outer linear measure is defined (§3) in such a way that Carathéodory's postulational theory of measure may be used.

Received November 13, 1939.

[1] Numbers in brackets refer to the bibliography at the end of the paper.