# A MODIFIED MOMENT PROBLEM IN TWO VARIABLES

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1. In the consideration of the moment problem

$$\int_0^1 \Phi(v) v^{\lambda_n} dv = Y_n \qquad (n = 1, 2, \cdots), \tag{1.1}$$

where $\{\lambda_n\}$ and $\{Y_n\}$ are given sequences of complex numbers with $\text{Re}(\lambda_n) > -\frac{1}{2}$ and $\Phi$ is sought as a function of $\mathfrak{L}^2$, two cases arise, namely, the homogeneous case and the non-homogeneous case (in the homogeneous case one naturally seeks a function $\Phi \not\equiv 0$). The homogeneous problem is of especial importance since its solution answers the question of the closure of the set $\{v^{\lambda_n}\}$ over $(0, 1)$, the set $\{v^{\lambda_n}\}$ being not closed or closed according as the system (1.1) with $Y_n = 0$ $(n = 1, 2, 3, \cdots)$ has or has not a solution $\Phi \not\equiv 0$, $\Phi \in \mathfrak{L}^2$. Two methods of attack have been used in the homogeneous case.[1] The one method, a real variable method, reduces the problem to an infinite system of linear equations in an infinite number of unknowns and the solution is obtained in terms of infinite determinants, which by use of Cauchy's theorem on determinants and other ingenious considerations yields the result that a necessary and sufficient condition for the closure of the set is the divergence of the series:

$$\sum_{n=1}^{\infty} g(\lambda_n), \tag{1.2}$$

where

$$g(\lambda) = \frac{1 + 2\,\text{Re}(\lambda)}{1 + |\lambda|^2}. \tag{1.3}$$

The other method of attack uses Fourier transforms in the complex plane to reduce the problem to a problem on the zeros of analytic functions of a certain class; again the same condition is obtained. Clearly both methods are applicable to the non-homogeneous case.

In this paper we shall be concerned with the moment problem in two variables,[2] namely,

$$\int_0^1 \int_0^1 \Phi(v_1, v_2) v_1^{\lambda_n} v_2^{\mu_n} dv_1\, dv_2 = Y_n \qquad (n = 1, 2, \cdots), \tag{1.4}$$

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[1] Cf. Müntz [1], Szász [1] for the first method and Paley-Wiener [2], p. 32, for the second method. (The numbers in brackets refer to the bibliography.) The method of attack of the present paper was suggested by a study of the work of Paley and Wiener. The authors are indebted to Professors Hildebrandt, Tamarkin, and Wiener for helpful advice.

[2] It will be clear that the methods are immediately applicable to the corresponding problem in $k$ variables, rather than in two.