A CONVERSE THEOREM CONCERNING THE DIAMETRAL LOCUS OF AN ALGEBRAIC CURVE

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1. **Introduction.** If a conic be cut by any system of parallel secants, the locus of the midpoint of the two intersections with each secant is a straight line, called a diameter of the conic. This diameter is the polar with respect to the conic of the common infinite point of the parallel secants.

It was proved by Sir Isaac Newton[1] that the preceding elementary property of conics extends to an algebraic curve A of any degree n. The intersection of A with any straight line l is a system of n real or imaginary points; let G denote their centroid. Then the locus of G as l moves parallel to itself is a straight line d, which may be called a *diameter* of A. The line d is, in fact, the linear polar with respect to A of the fixed infinite point of the system of parallel secants.

If any n curve-elements $\gamma_1, \gamma_2, \cdots, \gamma_n$ are cut by a system of parallel secants, we naturally define the corresponding "diametral locus" as the locus of the centroid G of the respective intersection-points $p_1, p_2, \cdots, p_n$ of the given curve-elements with an arbitrary secant l of the given parallel system. Suppose that this diametral locus is a straight line *for every system of parallel secants.* Then we shall prove in this paper (Theorem II) that the curve-elements γ_i must belong to the same algebraic curve of degree n, possibly a reducible one.

Let $\sigma_1, \sigma_2, \cdots, \sigma_n$ denote n hypersurface-elements in Euclidean space of any number $m + 1$ of dimensions. If the diametral locus of these hypersurface-elements relative to an arbitrary system of parallel secants is a hyperplane, then the elements must belong to an algebraic hypersurface of degree n, possibly reducible. This theorem (III), the natural analogue in higher dimensions of the one stated in the preceding paragraph, is proved in §6.

We conclude (§7) with a discussion of the conditions on the curve- and hypersurface-elements γ_i, σ_i under which our results are function-theoretically valid.

2. **Relation to other literature.** Our results include as very special cases certain converse theorems concerning polynomials, $y = P(x)$, recently given by Howard Levi.[2] This author states his result in an analytic form, which can be seen to amount geometrically to this: If the diametral locus, relative to an arbitrary direction, determined by n elements of an entire function $y = f(x)$ is a vertical straight line, then the entire function must be a polynomial. This

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[1] See Salmon, *Higher Plane Curves*, Dublin, 1873, p. 109.

[2] *On the values assumed by polynomials*, Bull. Amer. Math. Soc., vol. 45(1939), pp. 570–575.