INFINITE POWERS OF MATRICES AND CHARACTERISTIC ROOTS

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1. **Introduction.** Frazer, Duncan and Collar [2][1] have studied special cases of infinite powers of complex matrices. Necessary and sufficient conditions for the existence of such powers and the vanishing of such powers are given in the present paper. By the use of these powers one can solve one of the outstanding problems in applied mechanics [1, 6]. The results for vanishing infinite powers yield a simple proof of a theorem of Frobenius [3] to the effect that when the complex matrices are partially ordered in a certain way, the maximum absolute value of the characteristic roots of a complex matrix A remains the same or increases when A is increased. This result is useful in computing upper limits to the characteristic roots of a matrix, especially for real matrices with non-negative elements. Such matrices have been studied by various writers [4, 7].

2. **Definitions.** Let A be a square matrix with elements in the complex field. Let the elements in the i-th row and j-th column of A^n be denoted by $a_{ij}^{[n]}$. If for each pair of values of i and j the elements $a_{ij}^{[1]}, a_{ij}^{[2]}, a_{ij}^{[3]}, \cdots$ converge to a limit b_{ij} in the usual sense, we say that A^∞ *exists* and is the matrix (b_{ij}). Otherwise, we say that A^∞ does not exist. Let $A = (a_{ij})$ and $B = (b_{ij})$ have respectively non-negative and complex elements which satisfy $a_{ij} \geqq |b_{ij}|$. We say that A *contains* B, and write $A \supset B$. If $A \supset 0$, we term A *non-negative.* The smallest circle with center at the origin in the Argand diagram containing the characteristic roots of A is termed the *characteristic circle* of A. Let $[A]$ be the matrix whose elements are the absolute values of the corresponding elements of A. We term $[A]$ the *absolute matrix* of A.

3. **Existence of the infinite power of a matrix.** By means of the Jordan normal form[2] we shall prove the following

Theorem 1. *The infinite power of a complex matrix A exists if and only if the characteristic roots of A corresponding to elementary divisors (taken with respect to the complex field) of degree greater than* 1 *are in absolute value less than* 1, *and the remaining characteristic roots are equal to* 1 *or in absolute value less than* 1.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] L. E. Dickson, *Modern Algebraic Theories*, p. 106.