THE JACOBI CONDITION FOR UNILATERAL VARIATIONS

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The simplest type of parametric problem of the calculus of variations is that of minimizing an integral

$$J = \int_{t_1}^{t_2} F(x, y, x', y')\, dt$$

in a class of admissible curves

$$(1) \qquad x = x(t), \qquad y = y(t) \qquad (t_1 \leqq t \leqq t_2)$$

joining two fixed points 1 and 2 in the xy-plane.[1] We shall suppose that the integrand function $F(x, y, x', y')$ satisfies the usual continuity and homogeneity properties for (x, y) in a region R of the plane and for all $(x', y') \neq (0, 0)$.[2] A curve (1) will be called *admissible* if it lies in or on the boundary of a region R' within the region R and is of class[3] C$'$ except possibly at a finite number of corners.

A familiar formulation of the Jacobi condition states that there shall be no pair of conjugate points on any arc of a minimizing curve E_{12} which lies interior to the region R', is of class C$'$, and has $F_1 \neq 0$ along it. The well-known geometric proof of this condition by means of the Kneser envelope theorem of a one-parameter family of extremals does not necessarily apply to every extremal arc of E_{12} in the case when E_{12} may have arcs in common with the boundary of the region R' of admissible variations, and so far as the author knows, no use has yet been made of the second variation in this connection. There is no restriction on the length of the arcs of the minimizing curve in common with the boundary of R' which are not extremals as the sufficient conditions of Bliss[4] show. The purpose of this paper is to show that by means of the Kneser envelope theorem and two additional properties of the envelope we are able to complete the proof of the Jacobi condition for the situation just described.[5]

Suppose, for sake of illustration, that the minimizing curve E_{12} is an *extremal*

Received July 31, 1939; presented to the American Mathematical Society, December 29, 1939.

[1] We shall denote all derivatives with respect to the independent variable t by primes and all others by subscripts.

[2] See, e.g., O. Bolza, *Lectures on the Calculus of Variations*, University of Chicago, p. 117.

[3] For a definition of the term class as here used see Bolza, loc. cit., p. 116.

[4] G. A. Bliss, *Sufficient conditions for a minimum with respect to one-sided variations*, Transactions of the American Mathematical Society, vol. 5(1904), p. 491.

[5] Incidentally, the results of this paper are of historical interest since they eliminate the restriction, in the case of the geometric formulation, that the envelope have a regressive branch at its points of contact with the minimizing curve.