# SOME GENERALIZATIONS OF THE THEORY OF ORTHOGONAL POLYNOMIALS

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1. **Introduction.** If a function $\rho(x)$ is integrable and non-negative on an interval $(a, b)$ and is such that $\int_a^b \rho(x)\,dx > 0$, a set of polynomials $[p_n(x) = a_n x^n + b_n x^{n-1} + \cdots]$ is uniquely determined, except for a constant factor, by the relations

$$\int_a^b \rho(x) p_n(x) p_m(x)\,dx = 0, \qquad m \neq n. \tag{1}$$

The set of polynomials defined in this way is said to be orthogonal with respect to the weight function $\rho(x)$ over the interval $(a, b)$.

For the weight functions

$$\begin{aligned} &\rho(x) = (x-a)^\alpha (b-x)^\beta, && \alpha > -1, \beta > -1, \\ &\rho(x) = (x-a)^\alpha e^{-\beta x}, && \alpha > -1, \beta > 0, b = \infty, \\ &\rho(x) = e^{-\alpha x^2 + \beta x}, && \alpha > 0, a = -\infty, b = \infty, \end{aligned} \tag{a}$$

the polynomials $[p_n(x)]$ are respectively those of Jacobi, Laguerre, and Hermite. Each of these weight functions satisfies the Pearson differential equation

$$\frac{1}{\rho(x)} \frac{d}{dx} \rho(x) = \frac{Ax + B}{Cx^2 + Dx + E} \equiv \frac{Ax + B}{M(x)}, \tag{2}$$

if $A, B, C, D, E$ are given suitable values.

The class of functions defined by (2) when $A$, $B$, $C$, $D$, $E$ range through all real values ($Cx^2 + Dx + E \not\equiv 0$) is such that for each non-identically vanishing member $\rho(x)$, the expression $[\rho(x)]^{-1} d^n [M^n(x) \rho(x)]/dx^n$, where $M^n(x)$ means $[M(x)]^n$, is a polynomial in $x$ of degree $n$ at most. The set of polynomials

$$q_0(x) = 1, \qquad q_n(x) = \frac{1}{\rho(x)} \frac{d^n}{dx^n} [M^n(x) \rho(x)] \qquad (n = 1, 2, 3, \cdots) \tag{3}$$

satisfies (1) when $\rho(x)$ is one of the functions (a). In other cases the property of orthogonality is lost because $\rho(x)$ is such that the integral (1) ceases to have a meaning. The corresponding sets of polynomials (3) will by way of distinction be called non-orthogonal.

As is known, each of the systems of orthogonal polynomials satisfies a recursion formula and a Christoffel-Darboux identity deduced from the recursion formula, and has the property of representing suitable functions by means of convergent

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