THEOREMS OF THE PICARD TYPE

By Olaf Helmer

1. **Introduction.** We shall denote the *order* and the *exponent of convergence* of an integral function $f(z)$ by

$$(1) \qquad \rho = \operatorname{ord} f \qquad \text{and} \qquad \alpha = \exp f,$$

respectively. It is known that the latter does not exceed the former and is usually equal to it. In the case where

$$(2) \qquad \exp f < \operatorname{ord} f$$

occurs, we shall say that the function $f(z)$ is *exceptional*.

Picard's original theorem that an integral function $f(z)$ does not omit more than one value has been replaced by the stronger theorem of Picard-Borel which, in the above terminology, can be formulated as follows: For an integral function $f(z)$ there is at most one constant c for which the function $f(z) - c$ is exceptional.

This theorem has, on the one hand, been generalized so as to apply to meromorphic functions.[1] On the other, it has been refined in various directions; in particular, the constant c has been replaced by a polynomial,[2] and even by an integral function whose order is smaller than that of $f(z)$.[3] It is the main object of this paper to prove another similar generalization of the Picard-Borel theorem that goes a little further. We shall consider pairs of integral functions $f(z)$, $g(z)$, and it will be our object to inquire into the number of integral functions $A(z)$ whose order is less than the larger of the orders of $f(z)$ and $g(z)$, for which the function $f(z) + A(z) \cdot g(z)$ is exceptional. In the special case where $g(z) = 1$ we obtain the previously considered cases cited above.

Of the theorems leading up to these results, the first is a precise statement concerning the order of the product of two integral functions, while the second and third deal with the order of an exponential form $A \cdot e^F + B \cdot e^G$, where A, B are integral functions and F, G are polynomials. These two latter theorems are not new, but have been proved in what follows for reasons of completeness.[4]

Throughout this paper we shall restrict ourselves to integral functions of *finite* order.

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[1] Cf. R. Nevanlinna, *Le Théorème de Picard-Borel*, Paris, 1929.

[2] Cf. E. Borel, *Leçons sur les Fonctions Entières*, 2d edition, Paris, 1921, p. 89.

[3] Cf. G. Valiron, *General Theory of Integral Functions*, Toulouse, 1923, p. 303.

[4] Theorem 3 has been used by Borel, but the proof he gives contains a mistake (see page 101 of reference in footnote 2). A correct proof can be found in G. Vivanti, *Theorie der eindeutigen analytischen Funktionen*, Leipzig, 1906.