THE ERGODIC THEOREM FOR A SEQUENCE OF FUNCTIONS

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1. Introduction. The Birkhoff ergodic theorem,\(^1\) for a single measure-preserving transformation \(T(P)\) on a space \(\Omega\) with a measure\(^2\) \(m\) and a function \(f(P)\) in \(L_1(\Omega)\), states that

\[
f^*(P) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n P)
\]

exists a.e.\(^3\) in \(\Omega\). In the case of a flow (i.e., a one-parameter family of measure-preserving transformations \(T_t\) for which \(T_t \cdot T_s = T_{t+s}\) \((-\infty < t < \infty, -\infty < s < \infty\)), and for which, if \(A\) is measurable in \(\Omega\), the set of \((P, t)\) points such that \(T_t P \in A\) is measurable in the space \(\Omega \times t\), the theorem states that

\[
\lim_{t \to \infty} \int_0^T f(P_t) \, dt \text{ exists for almost all } P \in \Omega.
\]

The object of this paper is to extend the theorem to a dominated, convergent, double sequence of functions \([f_{mn}(P)]\) of \(L_1(\Omega)\) for which \(\lim_{m, n \to \infty} f_{mn}(P) = f_0(P)\), to obtain a more general ergodic theorem and a condition for \(\lim_{m, n \to \infty} f_{mn}^*(P) = f_0^*(P)\). When \(f_{mn}(P)\) is the characteristic function of a set \(E_{mn}\) and \(T(P)\) is metrically transitive, the theorems allow the usual interpretation: that certain time means may be replaced by space means.

We assume throughout that \(\Omega\) has a measure \(m\) defined on it and that \(m(\Omega)\) is finite.

2. The discrete case.

**Theorem 1.** If the complex-valued functions \(f_{mn}(P)\) \((m, n = 1, 2, \ldots)\) and the positive function \(\phi(P)\) are summable over \(\Omega\) with \(|f_{mn}(P)| < \phi(P)\), and \(\lim_{m, n \to \infty} f_{mn}(P) = f_0(P)\) a.e. in \(\Omega\), then for any sequences of positive integers \([m_{ij}]\) and \([n_{ij}]\) for which \(\lim_{i, j \to \infty} m_{ij} = \lim_{i, j \to \infty} n_{ij} = \infty\)

\[
\lim_{i, j \to \infty} \frac{1}{i} \sum_{j=0}^{i-1} f_{m_{ij}n_{ij}}(T^j P) = f_0^*(P),
\]

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\(^2\) A discussion of an abstract space with measure \(m\) is to be found in Hopf, loc. cit., p. 1.

\(^3\) The abbreviation a.e. means almost everywhere.

\(^4\) \(T_t P\) will be written as \(P_t\).