# SOME SUMS INVOLVING POLYNOMIALS IN A GALOIS FIELD

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1. Let $GF(p^n)$ denote a Galois (finite) field of order $p^n$. Let

$$M = M(x) = c_0x^m + c_1x^{m-1} + \cdots + c_m$$

denote a polynomial in the indeterminate $x$ with coefficients in the $GF(p^n)$. For $c_0 = 1$, $M$ is said to be primary; provided $c_0 \neq 0$, we write $\deg M = m$. In this note we evaluate certain sums extended over the set of primary polynomials of a given degree. Some of the formulas are new, others were proved earlier but are derived here in a more direct manner. In particular we shall prove the identities

$$\sum_{\deg M=m} M^{p^{n(m+k)}-1} = (-1)^m \frac{F_{m+k}}{L_m F_k^{p^{nm}}}, \tag{1.1}$$

and

$$\sum_{\deg M=m}^{m} \frac{1}{M^{p^{nk}-1}} = \frac{L_{k+m-1}}{L_{k-1}L_m^{p^{nk}}}, \tag{1.2}$$

where

$$\begin{aligned} F_k &= (x^{p^{nk}} - x)(x^{p^{nk}} - x^{p^n}) \cdots (x^{p^{nk}} - x^{p^{n(k-1)}}), \\ L_k &= (x^{p^{nk}} - x)(x^{p^{n(k-1)}} - x) \cdots (x^{p^n} - x), \\ F_0 &= L_0 = 1. \end{aligned} \tag{1.3}$$

2. We shall require a number of known formulas.[1] Put

$$\psi_m(t) = \sum_{j=0}^{m} (-1)^{m-j} \begin{bmatrix} m \\ j \end{bmatrix} t^{p^{nj}}, \tag{2.1}$$

where

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{F_m}{F_j L_{m-j}^{p^{nj}}}, \qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = \frac{F_m}{L_m}, \qquad \begin{bmatrix} m \\ m \end{bmatrix} = 1.$$

Then $\psi_m(G) = 0$ for all $G$ of degree less than $m$, while $\psi_m(M) = F_m$ for $M$ primary of degree equal to $m$; indeed we have the factorizations

$$\psi_m(t) = \prod_{\deg G<m} (t + G) \tag{2.2}$$

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[1] *On certain functions connected with polynomials in a Galois field*, this Journal, vol. 1(1935), pp. 137–168, p. 139. This paper will be cited as DJ.