CONTRIBUTIONS TO THE THEORY OF HERMITIAN SERIES

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Introduction

The present paper is devoted to a study of *Hermitian series in the complex domain.* We shall consider expansions of the form

$$f(z) = \sum_{n=0}^{\infty} f_n h_n(z), \tag{1}$$

where

$$h_n(z) = e^{-\frac{1}{2}z^2} H_n(z) = (-1)^n e^{\frac{1}{2}z^2} \frac{d^n}{dz^n} [e^{-z^2}], \tag{2}$$

z being a complex variable.[1] While such series have been rather thoroughly studied for real values of the variable,[2] no adequate discussion of the complex case seems to exist anywhere in the literature. Various aspects of this problem, such as *the domain of convergence, the presence or absence of singularities on the boundary of this domain, gap-theorems,* and *the relations of Hermitian series to associated Dirichlet and Fourier series,* will be discussed in this paper.

The only phase of the complex theory for which the author has been able to find a discussion in the literature is the *representation problem.* Here G. N. Watson [30] and O. Volk [28, 29] have found essentially equivalent sufficient but far from necessary conditions for the representability of a given analytic function by a Hermitian series. We shall not discuss this problem here. We shall solve this problem in the second paper of this series to appear in the Transactions of the American Mathematical Society.

The first problem that confronts us in the complex theory is the *domain of convergence* of a given Hermitian series. While it is well known to analysts working in this field that *the domain of convergence is a strip,* $-\tau < y < \tau$, no simple proof of this fact appears to be available in the literature. Such a proof is given in Chapter 2 of the present paper.

Any convergence proof must be based upon some estimate of *the asymptotic behavior of the Hermitian functions $h_n(z)$ for large values of n.* Such estimates

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[1] The definition of a Hermitian polynomial differs from one author to the next. I shall employ the notation of an earlier paper of mine [12] to which the reader is referred for the formal properties of Hermitian functions used in this paper. Numbers in brackets refer to the bibliography at the end of the present paper.

[2] A good summary of the theory of Hermitian series in the real case is to be found in Chapter 4 of the treatise of G. Vitali and G. Sansone [26]. The Tchebycheff-Hermite polynomials of these writers differ from those of the present paper by the factor $(-1)^n$.