GROUPS AND ABELIAN GROUPS IN TERMS OF NEGATIVE ADDITION AND NEGATION

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1. **Object.** Let $+$ be the "direct" operation in groups and a' the inverse, or "negative", of a. Let Δ be the operation of "negative addition", given by $a \Delta b = (a + b)'$. The object of this paper is to define groups and Abelian groups in terms of Δ, $'$.

The definitions are postulational. I give a set of independent conditions on a class K, a binary operation Δ, and a unary operation $'$, so that the system $(K, \Delta, ')$ is a group. Similarly for an Abelian group. The postulates are such that the set for groups is embraced in the set for Abelian groups. The use of the operation Δ and the use of a unary operation are, as far as I know, novel in the definition of groups and Abelian groups.[1]

The postulates for groups are P_0-P_5 below; those for Abelian groups are P_0-P_6. Of these, P_0 merely rules out trivial systems—systems in which K is empty or contains but a single element. For unrestricted groups, omit P_0. For merely non-vacuous groups, replace P_0 by: P_0'. *K is not empty.* The postulates and the theorems derived from the postulates will bring out properties of Δ not easily seen when $a \Delta b$ is written in the form $(a + b)'$.

2. **Postulates for groups.** The postulates for groups follow. In P_3-P_5, supply the clause: *whenever the elements involved and their combinations are in K.*

P_0. *K contains at least two distinct elements.*

P_1. *$a \Delta b$ is in K whenever a, b are in K.*

P_2. *a' is in K whenever a is in K.*

P_3. $(a \Delta b)' \Delta c = a \Delta (b \Delta c)'$.

P_4. $(a \Delta b) \Delta a = b$.

P_5. $(a \Delta b)' = b' \Delta a'$.

3. **Theorems. Sufficiency of P_0-P_5 for groups.** Theorems T_1-T_{13} following, derivable from P_0-P_5, will establish the sufficiency of P_0-P_5 for groups.

T_1. $b \Delta (a \Delta b) = a$.[2]

Received June 27, 1939; presented to the American Mathematical Society, April 15, 1939, under the title *Groups and Abelian groups in terms of negative addition.*

[1] For references to postulates for groups and Abelian groups in terms of operations other than the "direct" operation, see footnotes to my *Postulates for abelian groups and fields in terms of non-associative operations*, Trans. Amer. Math. Soc., vol. 43(1938), pp. 1–6. For the first explicit use of an undefined unary operation in a set of postulates, see my paper *Whitehead and Russell's theory of deduction as a mathematical science*, Bull. Amer. Math. Soc., vol. 37(1931), pp. 480-488. The term *unary* was introduced in this paper.

[2] T_1 can evidently be used in place of P_4 in the postulates for groups and Abelian groups.