THE JACOBI CONDITION FOR THE DOUBLE INTEGRAL PROBLEM OF THE CALCULUS OF VARIATIONS

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1. **Introduction.** This paper is concerned with the Jacobi condition for the problem of minimizing a double integral

$$I = \iint_A f(x, y, z, z_x, z_y)\, dx\, dy \tag{1.1}$$

in a class of admissible surfaces $z = z(x, y)$ with fixed values on the boundary C of A. If C is supposed to be a simply closed regular curve, the usual form of the Jacobi condition may be stated as follows: If A_0 is the interior of a simply closed curve C_0, and A_0 is a proper subset of A, then along a minimizing surface E of class C″ there can exist no solution $u(x, y)$ of the accessory (Jacobi) equation such that $u \equiv 0$ on C_0, $u \not\equiv 0$ on A_0, and $|u_x| + |u_y| \not\equiv 0$ on the part of C_0 lying in A. For a minimizing surface of class C′, Schuler [10][1] has stated the above result in terms of the Haar form of the accessory equation for the case in which the boundary C_0 of A_0 lies interior to A. It is to be remarked that for the more classical case in which the minimizing surface is supposed to be of class C″ the assumption $|u_x| + |u_y| \not\equiv 0$ on C_0 is extraneous if C_0 is interior to A and there exists an elementary solution of the Jacobi equation.[2]

The purpose of the present paper is to remove certain factors which are present in the above formulation of the Jacobi condition simply for convenience of proof. We shall allow A to be an arbitrary connected open set with frontier C. The "simply closed regular curve C_0 and its interior A_0, where A_0 is a proper subset of A" is replaced by "a connected open proper subset $\mathfrak{A}$ of A and its frontier $\mathfrak{C}$". Moreover, we do not suppose that the solution u of the accessory equation is defined on the complement of $\mathfrak{A} + \mathfrak{C}$, but simply that u is of class C′ in $\mathfrak{A}$ and such that u, u_x, and u_y approach continuous limit values on $\mathfrak{C}$. Since relatively little is known concerning the qualitative character of the solutions of the type of differential equations with which we are here concerned, this result is a desirable extension of the usual formulation. In §5 it is shown that the assumption $|u_x| + |u_y| \not\equiv 0$ on $\mathfrak{C}$ is extraneous in case $\mathfrak{C}$ belongs entirely to A and there exists an elementary solution of the accessory equation. Finally,

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[1] Numerals in square brackets refer to the bibliography at the end of this paper.

[2] See Bolza [2], pp. 676–679. It is readily seen that if the interior A_0 of the simply closed curve C_0 is a proper subset of A the proof given by Bolza on pp. 676–678 establishes the form of the Jacobi condition stated above, although Bolza assumes throughout that C_0 is interior to A.