ALGEBRAS DEFINED BY GROUPS WHOSE MEMBERS ARE OF THE FORM $A^x B^y$

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This paper is a study of a class of groups of not very complicated structure, considered to be algebras. To make an algebra of a group we must introduce the additive property giving the elements of the group coefficients from a field, in this case the complex field.[1] As an algebra a reducibility appears. It has long been known that an abstract group could be represented in several ways as a linear group, each such representation being practically a matrix, that is, a simple algebra, a quadrate algebra, now called a total matric algebra. It happens thus that a group as an algebra becomes semisimple. The object of the paper is to effect this reduction by simple means. The defining basal units of the algebra, originally called *vids* by C. S. Peirce—a name that should have been retained—, are found by first determining hypernumbers of the algebra which are the partial moduli of the separate integral subalgebras. In the general case of groups this involves solving algebraic equations, but for this class we do not need to do that.

This class of groups exemplifies almost every group property rather simply, and also many properties of algebras. In particular, when there is a central of the group different from ordinary unity, we can extend the field and reduce the order of the algebra. Interesting examples of some of the recent work in algebras can be formed.

I. The groups

The most elementary groups are the cyclic, defined by a single generator A with $A^n = 1$. We may take as next most elementary those defined by two generators A, B with every member of the form $A^x B^y$ and with the relations

$$A^a = 1 = B^b, \qquad A^i = B^j, \qquad BA = A^q B^r, \qquad ei \equiv 0\ (a), \qquad ej \equiv 0\ (b).$$

For such groups we have the

Theorem.

$$B^x A^y = A^{yq^x} B^{xr^y} \qquad (x = 1, 2, \cdots, a;\ y = 1, 2, \cdots, j);$$

$$q^c \equiv 1\ (a), \qquad r^d \equiv 1\ (b), \qquad a = dg, \qquad b = ch, \qquad q = 1 + \omega d, \qquad r = 1 + \mu c.$$

Synthetic proof. Cayley showed (Amer. Jour. Math., vol. 1(1878), pp. 174–176) that abstract groups could be represented by assigning to each generator a

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[1] See Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, 1934, pp. 167–168.