DUALITY AND COMMUTATIVITY OF GROUPS

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Two groups are duals of each other if there exists an anti-isomorphism between their lattices of subgroups. The existence of a dual implies that all the elements are of finite order. Not every group possesses a dual since there do not exist duals of Hamiltonian groups. To be a dual and to be Abelian are equivalent properties of finite groups generated by elements of prime order p; and this statement is but one in a larger class of theorems connecting duality and commutativity.

Abelian groups possess duals if, and only if, they are self-dual; and a necessary and sufficient condition for self-duality of an Abelian group is the absence of elements of infinite order together with the finiteness of its primary components. Thus the class of self-dual Abelian groups proves to be exactly the same as the class of those Abelian groups—determined in an earlier note[1]—which admit an operation mapping the subgroups upon isomorphic quotient-groups.

1. A *dualism* between the groups G and H is a one-one correspondence $\mathbf{d}$, mapping the set of all the subgroups of G upon the whole set of subgroups of H in such a way that

$$S \leqq T \text{ if, and only if, } T^{\mathbf{d}} \leqq S^{\mathbf{d}}.$$

Such a dualism maps the cross-cut (join-group) of a set of subgroups upon the join-group (cross-cut) of the set of the corresponding subgroups; and it maps in particular G upon the identity in H and the identity in G upon H. Two groups are *duals* of each other if there exists a dualism between them.

The inverse operation of a dualism is again a dualism between the same groups, whereas the product of two dualisms (if it exists) is a so-called *subgroup-isomorphism*,[2] as it maps a subgroup of a subgroup upon a subgroup of the corresponding subgroup.

If $\mathbf{d}$ is a dualism between the groups G and H, and if in particular $G = H$, then $\mathbf{d}$ is an *autodualism* of G and the group G is *self-dual.*

A dualism $\mathbf{d}$ between the groups G and H induces dualisms in all the quotient-groups of H and in all those subgroups of G which it maps upon normal subgroups of H. This principle will be used very often.

The following theorem is known[3] and may be stated for future reference.

Received March 17, 1939; presented to the American Mathematical Society, April 7–8, 1939.

[1] Reinhold Baer, *Dualism in Abelian groups*, Bull. Amer. Math. Soc., vol. 43(1937), pp. 121–124; cited as Dualism.

[2] Cf. R. Baer, Bull. Amer. Math. Soc., vol. 44(1938), pp. 817–820; Amer. Jour. of Math., vol. 61(1939), pp. 1–44; cited as Isomorphism.

[3] See Dualism.