CONFORMAL MAPPING OF MULTIPLY CONNECTED DOMAINS

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The theory of Plateau's and Douglas' problem furnishes powerful tools for obtaining theorems on conformal mapping. Douglas emphasized (1931) that Riemann's mapping theorem is a consequence of his solution of Plateau's problem; then he treated doubly connected domains and in a recent paper (1939) multiply connected domains.[1] With a different method I gave in a paper on Plateau's problem (1937) a proof of the theorem that every k-fold connected domain can be mapped conformally on a plane domain bounded by k circles.[2] The same method can be applied to the proof of the parallel-slit theorem[3] and, as will be shown in the thesis of Bella Manel, to mapping theorems for various other types of plane normal domains. It is the purpose of the present paper[4] first to give a simplification of the method by utilizing an integral introduced by Riemann in his doctoral thesis, and secondly, to prove a mapping theorem of a different character referring to normal domains which are Riemann surfaces with several sheets.

We consider a Riemann surface on a u, v-plane consisting of the interior of k unit circles which are connected in branch points of total multiplicity $2k - 2$; to this surface we affix $p \geqq 0$ full planes with two branch points each. Thus we define a class of domains B with the boundary b on the plane of $w = u + iv$. Now our theorem is: Each k-fold connected domain G in the x, y-plane with the boundary curves $g_1, g_2, \cdots, g_k$ (which we suppose to be rectifiable[5] Jordan curves) can be mapped conformally on a domain B of our class for any fixed p.

In this mapping the branch points on the full planes and one more branch point may be arbitrarily prescribed and, moreover, on each boundary circle b_ν of B a fixed point may be made to correspond to a fixed point of g_ν.

For the case $p = 0$, the theorem was stated by Riemann, according to oral tradition.[6] As is easily seen, the class of domains B depends on $3k - 6$ essential parameters for $k > 2$ and on one parameter for $k = 2$.

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[1] Annals of Mathematics, vol. 40(1939), pp. 205–298. This paper contains a complete bibliography of Douglas' previous publications on the Plateau problem.

[2] Annals of Mathematics, vol. 38(1937), p. 679 ff., here quoted as (A).

[3] See M. Shiffman's thesis *The Plateau problem for minimum surface of arbitrary topological structure* to appear in the American Journal of Mathematics.

[4] See also a note by the author in the Proceedings Nat. Acad. Sciences, vol. 24(1938), p. 519 ff. Furthermore, reference is made to a detailed paper by the author on Plateau's and Douglas' problem forthcoming in the Acta Mathematica, here quoted as (B).

[5] This assumption is not essential and can easily be dropped.

[6] See Bieberbach, Sitzungsberichte Berliner Math. Ges., vol. 24(1925), p. 6 ff., where a proof is indicated; and Grunsky, Sitzungsberichte Preuss. Akademie Wiss., 1937, p. 40 ff., where another proof is given.