FORMAL POWER SERIES TRANSFORMATIONS

By Daniel C. Lewis, Jr.

1. **Introduction and summary of principal results.** Consider an analytic non-singular transformation T of the neighborhood of the origin of an n-dimensional space into itself. The t-th iterate of T, denoted by T^t, is defined when t is an integer and may be represented by convergent power series whose coefficients are functions of t. Furthermore the relation $T^{t+\tau} = T^t T^\tau$ holds for all integral values of t and τ. The question which arises is this: Is it possible to define T^t for non-integral values of t so that this relation holds for all t and τ? If we require the dependence on t to be analytic, the answer is certainly in the negative (save for comparatively rare exceptions), as can be shown by examples ($n = 2$).[1] It is, however, in general, possible, as we prove in this paper, to define the coefficients of the series for T^t for all values of t, without regard to whether or not they converge, in such a manner that $T^{t+\tau} = T^t T^\tau$ holds in a purely formal sense. Furthermore, the coefficients may be defined as comparatively simple functions of t, being, in fact, polynomials in t and a finite number of expressions of the form $e^{\mu t}$. This result is important for dynamical theory, but we do not restrict ourselves here to the type of transformation arising in dynamics. The special properties of the dynamical transformations will be studied in a future paper.

Previous work in this field has been done by C. L. Bouton,[2] who treated the case when the matrix of coefficients of the linear terms is the unit matrix, and by G. D. Birkhoff,[3] who treated the so-called "conservative" surface transformations ($n = 2$) which arise in dynamics.

The case when T is linear is of special importance. Here questions of convergence have no place. The problem is essentially that of defining the t-th

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[1] If such convergent series exist, they must satisfy a system of differential equations of the form $dx_i/dt = X_i(x)$ in which the X_i are convergent series in $x_1, \cdots, x_n$ but independent of t. Cf. §6.2. In the case of "conservative" surface transformations ($n = 2$) these equations have an analytic first integral and are therefore completely integrable. But examples can be constructed of conservative transformations of "stable type" which in this sense cannot be integrable. Cf., for example, G. D. Birkhoff, *Dynamical Systems*, American Mathematical Society Colloquium Publications, vol. 9, New York, 1927, p. 259. The general solutions of the "non-integrable" equations there discussed yield such transformations when the independent variable takes on the value 2π. For further information on this subject cf. G. D. Birkhoff, *Surface transformations and their dynamical applications*, Acta Mathematica, vol. 43(1922), pp. 1–119, especially p. 16.

[2] Bulletin of the American Mathematical Society, vol. 23(1916), p. 73.

[3] Acta Mathematica, loc. cit. The restriction to conservative transformations is not entirely essential.