THE FIRST CANONICAL PENCIL

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I. Introduction

Among the most important covariant lines which lie in the tangent plane to a surface S at a point P_x are the first canonical edge of Green [5],[1] the first directrix of Wilczynski [7], the reciprocal [5] with respect to the surface S of the projective normal [4], and the reciprocal with respect to S of the axis of Čech [3]. In view of the fact that these covariant lines, each of which was discovered by a different author, were characterized by apparently unrelated properties, it has been considered remarkable that they all should pass through a common point of the tangent plane. This point has been called the *canonical point.* Wilczynski [7] and Green [5] have referred to lines in the tangent plane to S at P_x as lines of the *first kind.* Accordingly, a covariant line which passes through the *canonical point* has been called a *canonical line of the first kind.* The totality of canonical lines of the first kind form the *first canonical pencil* [1]. The primary purpose of the author in this note is to present a new geometric characterization of a general canonical line of the first kind. For this purpose the projective normal is first constructed in a new way.

II. The projective normal

Let the surface S be referred to its asymptotic net as parametric, and let us choose the associated fundamental differential equations in *Fubini's canonical form*

$$(1)\qquad \begin{cases} x_{uu} = px + \theta_u x_u + \beta x_v, \\ x_{vv} = qx + \gamma x_u + \theta_v x_v, \end{cases}$$

where $\theta = \log \beta\gamma$. Let l denote an arbitrarily chosen line of the first kind. The line l therefore intersects the u- and v-tangents to S at P_x in points ρ and σ whose general coördinates are of the forms $\rho = x_u - bx$, $\sigma = x_v - ax$, in which a and b are functions of u and v. Let l' denote the reciprocal of l with respect to S at P_x. Let τ and ω denote, respectively, the points distinct from P_x in which the line l' intersects the quadrics of Wilczynski and Lie [7] at the point P_x. The general coördinates of τ and ω may easily be found to be given by the expressions $\tau = z + (ab - \frac{1}{2}\theta_{uv})x$ and $\omega = \tau - \frac{1}{2}(\beta\gamma)x$, in which $z = x_{uv} - ax_u - bx_v$. For this purpose one would make use of the equations

$$(2)\qquad 2(x_2x_3 - x_1x_4) - \theta_{uv}x_4^2 = 0$$

Received December 15, 1938; presented to the American Mathematical Society, April 9, 1937.

[1] Numbers in brackets refer to the bibliography at the end of the paper.