CAYLEY NUMBERS AND NORMAL SIMPLE LIE ALGEBRAS OF TYPE G

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In an earlier paper[1] we discussed the set $\mathfrak{D}(\mathfrak{A})$ of derivations in an arbitrary algebra $\mathfrak{A}$ (not necessarily associative), i.e., the operators D in $\mathfrak{A}$ such that

$$(x + y)D = xD + yD, \qquad (x\alpha)D = (xD)\alpha, \qquad (xy)D = x(yD) + (xD)y,$$

α being in the underlying field Φ. We noted that $\mathfrak{D}$ is closed with respect to addition, scalar multiplication and commutation $[D, E] \equiv DE - ED$. Hence $\mathfrak{D}$ is a Lie algebra over Φ. We shall show here that if $\mathfrak{A}$ is a generalized Cayley algebra and Φ is of characteristic 0, then $\mathfrak{D}$ is normal simple of type G and all such Lie algebras may be obtained in this way. The derivation algebras are isomorphic if and only if the Cayley systems are. If Φ is algebraically closed, these results have been indicated by Cartan.[2] The extension to the general case given here depends essentially on the determination of the automorphisms of $\mathfrak{D}$ in the algebraically closed case. The structure of Cayley systems has been obtained by Zorn.[3] We give several extensions of his theory.

We require below the theorem that if $\mathfrak{A}_P$ is the algebra obtained by extending Φ to P, then $\mathfrak{D}(\mathfrak{A}_P) = \mathfrak{D}_P$.[4] We note also that if S is either an automorphism or anti-automorphism in $\mathfrak{A}$ such that $(x\alpha)S = (xS)\alpha^s$, where $\alpha \to \alpha^s$ is an automorphism in Φ, then $S^{-1}DS$ is a derivation for every D in $\mathfrak{D}$. If $\alpha^s \equiv \alpha$, $D \to S^{-1}DS$ is an automorphism of $\mathfrak{D}$ over Φ.

1. Let Q be a (generalized) quaternion algebra over a field of characteristic $\neq 2$. We do not exclude the possibility that $Q = \Phi_2$, the 2-rowed matrix algebra. A (generalized) Cayley algebra $\mathfrak{A}$ is a vector space of order 2 over Q, $\mathfrak{A} = Q1 + Qe_4$, in which

$$(a + be_4)(c + de_4) = (ac + \bar{d}b\alpha_4) + (da + b\bar{c})e_4, \tag{1}$$

where $\alpha_4 \neq 0$. If Q has basis $(1, e_1, e_2, e_3)$ such that $e_1^2 = \alpha_1$, $e_2^2 = \alpha_2$, $\alpha_i \neq 0$, $e_1e_2 = -e_2e_1 = e_3$ and $e_4^2 = \alpha_4$, then $1, e_1, \cdots, e_7$ is a basis for $\mathfrak{A}$ if $e_5 = e_1e_4$, $e_6 = e_4e_2$, $e_7 = e_3e_4$. $(1, e_1, e_4, e_5)$, $(1, e_4, e_2, e_6)$, $(1, e_3, e_4, e_7)$, $(1, e_1, e_6, e_7)$, $(1, e_2, e_5, e_7)$ and $(1, e_3, e_5, -e_6\alpha_1)$ are quaternion algebras. If e_i, e_j, e_k do not belong to one of these algebras, then $(e_ie_j)e_k = -e_i(e_je_k)$. It is sometimes

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[1] *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc., vol. 42(1937), pp. 206–224.

[2] *Les groupes rèels simples et continus*, Ann. de l'École Normale, vol. 31(1914), p. 298.

[3] *Alternativkörper und quadratische Systeme*, Hamb. Abhandl., vol. 9(1933), pp. 395–402.

[4] Loc. cit. (footnote 1), p. 213.