FUNCTIONS OF BOUNDED VARIATION AND NON-ABSOLUTELY CONVERGENT INTEGRALS IN TWO OR MORE DIMENSIONS

By R. L. Jeffery

1. **Introduction.** This paper originated in an attempt to extend to two or more dimensions a property which completely characterizes a function of bounded variation in one dimension. It has been shown elsewhere[1] that a necessary and sufficient condition that a function $F(x)$ be of bounded variation is that there exist a summable function $f(x)$ and a sequence of summable functions $s_n(x)$ tending to $f(x)$ with $\int_e s_n\,dx$ bounded in n and e for which

$$F(x) = \lim_{n\to\infty} \int_a^x s_n\,dx. \tag{1}$$

We carry this idea over to a function $F(x, y)$ of the two real variables x and y by saying that $F(x, y)$ is in class V_1 on the rectangle $R = (0, 0; a, b)$ if there exist a single-valued function $f(x, y)$ and a sequence of summable functions $s_n(x, y)$ tending to $f(x, y)$ such that $\int_e s_n\,dx\,dy$ is bounded in n and e, and for which

$$F(x, y) = \lim_{n\to\infty} \int_0^y \int_0^x s_n\,dx\,dy. \tag{2}$$

The corresponding statement for functions of more than two variables is obvious. Since the function $f(x, y)$ is the limit of a sequence of measurable functions, it is measurable, and it will be shown that $f(x, y)$ is summable. Hence (2) is a direct extension of (1). It will also be shown that a function in class V_1 on R has what have come to be called bounded variation properties: It can be represented as the difference of two non-decreasing[2] functions; consequently the set of discontinuities of $F(x, y)$ has zero measure, and the surface $z = F(x, y)$ has a tangent plane almost everywhere; if ω is a rectangular interval with sides parallel to the coördinate axes, then the function of intervals $F(\omega)$ associated with $F(x, y)$ by the relation $F(\omega) = \lim \int_\omega s_n\,dx\,dy$ is additive,[3] of bounded variation, and

Received December 10, 1938; in revised form, October 11, 1939.

[1] Jeffery, *Functions defined by a sequence of integrals and the inversion of approximate derived numbers*, Transactions of the American Mathematical Society, vol. 41(1937), p. 175, Theorem IV.

[2] Non-decreasing in the sense of Hobson. If $x_2 \geqq x_1$, $y_2 \geqq y_1$, then $F(x_2, y_2) \geqq F(x_1, y_1)$.

[3] A function of intervals $F(\omega)$ is additive on R if $F(\omega_1 + \omega_2) = F(\omega_1) + F(\omega_2)$ whenever ω_1, ω_2, and $\omega_1 + \omega_2$ are intervals on R. A function of sets $F(e)$ is additive if for every pair of disjunct sets e_1, e_2, $F(e_1 + e_2) = F(e_1) + F(e_2)$. $F(e)$ is completely additive if for every infinite sequence e_1, e_2, $\cdots$ of disjunct sets $F(\Sigma\, e_i) = \Sigma\, F(e_i)$.