A MATRIX THEORY OF n-DIMENSIONAL MEASUREMENT

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1. **Introduction.** In 1933 A. H. Copeland[1] gave a precise statement of the fundamental assumptions of a theory of measurement. It would serve for a one-dimensional theory of probability measurement. It is our purpose here to extend some of his results to n dimensions, thereby simplifying the treatment of n-dimensional probability theory.

With Copeland, a matrix of the form $\mathbf{x} = x^{(1)}, x^{(2)}, \cdots, x^{(k)}, \cdots$, where the k-th term $x^{(k)}$ is an arbitrary number, will be called a *variate*. If $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ is a set of n variates, and if $G(s_1, s_2, \cdots, s_n)$ is an arbitrary function of the variables $s_1, s_2, \cdots, s_n$, then $G(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ denotes a variate defined as follows: $G(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = G(x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}), G(x_1^{(2)}, x_2^{(2)}, \cdots, x_n^{(2)}), \cdots$. We shall define the probable value of the matrix $\mathbf{x}$ as

$$\mathbf{p}(\mathbf{x}) = \lim_{n \to \infty} \mathbf{p}_n(\mathbf{x}), \quad \text{where} \quad \mathbf{p}_n(\mathbf{x}) = \sum_{k=1}^{n} \frac{x^{(k)}}{n}.$$

Sometimes, however, this value $\mathbf{p}(\mathbf{x})$ may not exist, as is obvious.

In a similar manner we shall understand by an *n-dimensional variate* a variate defined by the matrix

$$(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n) = (x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}), (x_1^{(2)}, x_2^{(2)}, \cdots, x_n^{(2)}), \cdots$$

in which the k-th term $(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$ designates a point in an n-dimensional space. Further we will understand by $\phi_E(s_1, s_2, \cdots, s_n)$ the fundamental function of the point set E, and this function will have the value 1 or 0 according as the point $(s_1, s_2, \cdots, s_n)$ is a point of E or not. Then

$$\mathbf{p}[\phi_E(\mathbf{x}_1, \cdots, \mathbf{x}_n)] = \lim_{n \to \infty} \mathbf{p}_n[\phi_E(\mathbf{x}_1, \cdots, \mathbf{x}_n)],$$

where $\mathbf{p}_n[\phi_E(\mathbf{x}_1, \cdots, \mathbf{x}_n)]$ is defined as above. Hence if we let

$$\mathbf{p}[\phi_E(\mathbf{x}_1, \cdots, \mathbf{x}_n) = F(S_1, S_2, \cdots, S_n)$$

in which E is the n-dimensional cell defined as follows: $-\infty \leqq s_i \leqq S_i$ ($i = 1, 2, \cdots, n$), then $F(S_1, S_2, \cdots, S_n)$ will be the probability that the so-called digits of the matrix $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ will designate a point in the cell E. This function $F(s_1, s_2, \cdots, s_n)$ will be called the *n-dimensional accumulative proba-*

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[1] *A matrix theory of measurement*, Math. Zeit., vol. 37(1933), pp. 542–555.