ADDITIVITY AND CONTINUITY OF PERSPECTIVITY

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Introduction. If L is the set of linear subspaces of a projective geometry and $D(a)$ is the common dimension function[1] defined for the elements of L, it is easily shown that

$$(1) \qquad D(a + b) + D(ab) = D(a) + D(b) \qquad \text{for all } a, b \text{ in } L.$$

But if $D(a)$ is defined[2] to be the common dimension of a plus 1, then $D(a)$ satisfies not only (1) but also[3] $D(0) = 0$ and hence

$$(2) \qquad D(a + b) = D(a) + D(b) \qquad \text{if } ab = 0.$$

More generally, it will follow that for every finite m

$$(3) \qquad D\Big(\sum_{r=1}^{m} a_r\Big) = \sum_{r=1}^{m} D(a_r)$$

if $a_1, \cdots, a_m$ are linearly independent. Since in a given projective geometry there is a finite upper bound for the number of elements which can all be different from 0 and be linearly independent, there is no point in considering (3) for non-finite sums.

J. von Neumann has given a remarkable set of axioms defining a class of geometries which includes all projective geometries as well as a new type of geometry which he has named continuous geometry.[4] He has shown, too, that in each of these geometries there exist a dimension function $D(a)$ and a concept of independence such that (3) holds for all sets of independent elements whether their number is finite or not. And in continuous geometries there do exist countable sets of elements which are all different from 0 and are independent, but there are no such non-countable sets. In all of these geometries two elements a, b have the same dimension if and only if a can be mapped into b by a perspectivity (denoted by the symbols $a \sim b$).[5]

Among the axioms given by von Neumann to define projective and con-

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[1] That is, dimension of a point = 0, dimension of a line = 1, etc. The subspace of L which does not contain any points has common dimension = -1.

[2] It would be even more suggestive, from the point of continuous geometries, to define $D(a)$ to be [(common dimension of a) + 1]/[(common dimension of L) + 1].

[3] The same symbol will be used to denote both the number 0 and the empty subset of L, but there should be no confusion.

[4] See [3], [4], [5]. (Numbers in brackets refer to the bibliography at the end of the paper.)

[5] The actual construction of the dimension function is closely related to this property.