A DIFFERENTIAL EQUATION FOR ORTHOGONAL POLYNOMIALS

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Introduction. Any sequence of orthogonal polynomials[1] (OP) $\{\Phi_n(x)\}$ satisfies, as is known, a linear homogeneous difference equation of second order

$$(1) \qquad \Phi_n(x) - (x - c_n)\Phi_{n-1}(x) + \lambda_n\Phi_{n-2}(x) = 0 \qquad (n \geqq 2; \Phi_0 = 1, \Phi_1 = x - c_1),$$

where λ_n, c_n are constants, $\lambda_n > 0$. On the other hand, the classical OP of Hermite, Laguerre and Jacobi (special cases: Legendre's and trigonometrical polynomials) satisfy, in addition, a homogeneous linear differential equation of the following type (M, p. 33):

$$(2) \qquad A\Phi_n''(x) + B\Phi_n'(x) + C_n\Phi_n(x) = 0,$$

where A, B are polynomials in x, independent of n, of degrees not exceeding 2 and 1, respectively, and C_n is a constant depending on n.

The importance of differential equations in the study of OP needs no further emphasis. Thus, it is natural to seek to find other classes of OP for which a differential equation of this type exists, namely:

$$(3) \qquad A_n\Phi_n''(x) + B_n\Phi_n'(x) + C_n\Phi_n(x) = 0,$$

where A_n, B_n, C_n are polynomials in x, *each of fixed degree independent of* n, with coefficients eventually depending on n. In a note in the Comptes Rendus[2] the author has shown the existence of (3) for a certain general class of OP. The method employed, following Laguerre, yields rather an "existence proof" and is not readily applicable to the actual construction of the polynomials A_n, B_n, C_n.

The object of the present paper is to develop a new simple method for the effective construction of the differential equation (3) for an extended class of OP, of the same general type as in the note just cited. In application to the classical and other OP, this method yields, as by-products, many of their properties, old and new.

The method used is of a very elementary character.

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[1] $\Phi_n(x) = x^n - S_nx^{n-1} + d_{n,n-2}x^{n-2} + \cdots$; $\{\varphi_n(x) \equiv a_n\Phi_n(x)\}$ is the corresponding normalized sequence. The notations used are those of my monograph: *Théorie générale des polynomes orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, Fasc. 66, 1934 (hereafter designated by M), to which the reader is referred for further details.

[2] Jacques Chokhate, *Sur une classe étendue de fractions continues algébriques et sur les polynomes de Tchebycheff correspondants*, Comptes Rendus, vol. 191 (1930), pp. 989–990.