FUNCTIONS WHICH ASSUME RATIONAL VALUES AT RATIONAL POINTS

By J. W. Green

1. **Introduction**. Of the continuous functions which assume rational values for rational values of the argument, the familiar examples either are extremely regular, as the rational or piecewise rational functions, or else exhibit some extremely irregular properties. For example, the functions $x(t)$, $y(t)$ defining Peano's area filling curve assume rational values for rational t and are nowhere derivable. Likewise the familiar function defined with respect to Cantor's ternary set as $\frac{1}{2}$ on the middle extracted one-third, $\frac{1}{4}$ and $\frac{3}{4}$ on the extracted middle one-thirds of the left- and right-hand remaining intervals, respectively, etc. can be shown to assume rational values for rational x. This function possesses a derivative at no point except points where it is piecewise rational.

It is desired, then, to investigate what kinds of functions may possess the property of assuming rational values at rational points, which property we shall denote by (A). For example, it might be asked whether the only analytic functions with property (A) are the rational functions, or it might be asked if it is possible that a function exist with property (A), the function being analytic in no interval and yet, say, having a continuous derivative.

The author has been informed that something of this nature was discussed in a conversation between Weierstrass and Hilbert and that Hilbert exhibited an example; however, no record of the conversation seems to be extant. Neither has the author been able to find the example mentioned in the literature, or to discover its exact nature. Consequently, some of the results obtainable may be of sufficient interest to warrant their exposition.

2. **Analytic functions with property** (A). Let there be given any set E of rational points in the complex plane; that is, a set of points of the form $a + bi$, when a and b are rational. Let

$$z_1, z_2, \cdots$$

be an arrangement in denumerable order of the points of the set. Let z_0 be any other point of the complex plane. We prove the following lemma.

Lemma. *There exists an entire function of z assuming rational values at points of E and assuming a transcendental value at z_0.*

Let

$$P_1(z) = (z - z_1)/N_1,$$

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