LINEAL ELEMENT TRANSFORMATIONS OF SPACE FOR WHICH NORMAL CONGRUENCES OF CURVES ARE CONVERTED INTO NORMAL CONGRUENCES

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A doubly infinite family of curves in space for which orthogonal surfaces can be constructed is called a *normal congruence*. In this paper all transformations of lineal elements (x, y, z, y', z') of space are determined such that every normal congruence of curves shall be converted into a normal congruence. The infinite group obtained is shown to be isomorphic with the group of contact transformations in space of planar or surface elements (x, y, z, P, Q). (The only transformations in the new group which convert curves into curves are the conformal transformations, which form a ten-parameter group.) Our result may also be stated in this form: the only transformations which carry every pair of partial differential equations in involution into a pair of partial differential equations in involution are the contact transformations. That is, if every set of ∞^3 planar elements which are obtained from a set of ∞^1 surfaces is sent into a set of the same kind, then necessarily every single union is converted into a union.

We thus obtain a new characterization of the contact group in space. We do not *assume* that the individual surfaces in the family of ∞^1 surfaces are converted into surfaces. But from our complicated proof it does result that if *every* integrable field becomes such a field, then the individual unions are actually converted into individual unions; and *therefore* the result is a contact transformation.

A lineal element E is usually defined by the coördinates (x, y, z, y', z'), where (x, y, z) are the Cartesian coördinates of the point of the element E and $(1, y', z')$ are the direction numbers of the direction of the element E. From this it is of course obvious that in the case where ∞^1 lineal elements form a curve (or union) $y' = dy/dx$ and $z' = dz/dx$. Thus y' is the total derivative of y with respect to x and z' is the total derivative of z with respect to x. But in our work it will be more convenient to define an element E by the coördinates (x, y, z, p, q), where (x, y, z) are the Cartesian coördinates of the point of the element E and $(p, q, -1)$ are the direction numbers of the direction of the element E. From this it is seen that in the case where ∞^1 lineal elements form a curve (or union) $p = -dx/dz$ and $q = -dy/dz$. Thus p is minus the total derivative of x with respect to z and q is minus the total derivative of y with respect to z. The relationships between the old and new coördinate systems are obviously $p =$

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