SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF A CONTINUED FRACTION

By Walter Leighton

1. **A new convergence criterion.** Continued fractions of the form

$$1 + \frac{a_1}{1+} \frac{a_2}{1+} \cdots \qquad (a_n \neq 0), \tag{1.1}$$

where the quantities a_n are arbitrary complex numbers, are of particular importance from a function-theoretic point of view.[1] J. Worpitsky, E. B. Van Vleck, and A. Pringsheim proved independently that the conditions $| a_n | \leqq \frac{1}{4}$ $(n = 2, 3, \cdots)$ are sufficient to insure the convergence of (1.1).[2] O. Szász [2] showed that $\frac{1}{4}$ was the best such constant by pointing out that the continued fraction (1.1) with

$$a_n = -\tfrac{1}{4} - \epsilon \qquad (n = 1, 2, 3, \cdots)$$

diverges for every $\epsilon > 0$. Szász [1] proved that a sufficient condition for the convergence of (1.1) is

$$\sum_{n=2}^{\infty} | a_n | - \sum_{n=2}^{\infty} R(a_n) < 2,$$

where $R(a_n)$ is the real part of a_n. Leighton and Wall [1] proved that the conditions $| a_{2n+1} | \leqq \frac{1}{4}$, $| a_{2n} | \geqq \frac{25}{4}$ $(n = 1, 2, 3, \cdots)$ are sufficient. All the above conditions require at least an infinite subsequence of the numbers a_n to be $\leqq \frac{1}{4}$ in absolute value. The following theorem removes this condition in a rather unexpected manner.

Theorem. *If the numbers a_n satisfy the conditions*

$$| 1 + a_2 | \geqq | a_1 | + 1, \qquad | a_3 | \geqq \frac{2 + m}{1 - m},$$

$$| a_{2n} | \leqq m \qquad (n = 2, 3, 4, \cdots),$$

$$| a_{2n+1} | \geqq 2 + m + m \, | a_{2n-1} | \qquad (n = 2, 3, 4, \cdots),$$

where m is any positive number <1, the continued fraction (1.1) *converges.*

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[1] See, for example, Perron [2], Chapter VIII. (Numbers in brackets refer to the bibliography at the end of the paper.) W. T. Scott, in preparing a Rice Institute thesis, has found recently a number of results which strengthen significantly a natural generalization (Leighton [1]) of the material discussed by Perron (loc. cit.).

[2] O. Szász [2] discusses the history of this criterion.