PRÜFER IDEALS IN COMMUTATIVE RINGS

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1. **Introduction.** H. Prüfer has given[1] a general definition of an ideal in a field and has investigated the properties of these ideals in certain ideal systems. In the present paper a similar study is made, but the algebraic domain of reference will be taken to be a commutative ring $\mathfrak{R}$ having a unit element and possessing no divisors of zero.[2]

2. **Divisibility properties of elements.** The present section, although of some interest, is largely irrelevant to the main matter of the paper but can be conveniently treated at this point.

Let $\mathfrak{g}$ be a subring of $\mathfrak{R}$ with a unit element; the concept of divisibility can now be defined *relative to* $\mathfrak{g}$ so that the elements of $\mathfrak{g}$ may be thought of as the *integral elements* of $\mathfrak{R}$. If $a \neq 0$ and $b \neq 0$ are elements of $\mathfrak{g}$, then *a is divisible by b* if $a = bc$, where c is in $\mathfrak{g}$. Obviously, divisibility relative to $\mathfrak{g}$ is a reflexive and transitive property. If a and b divide each other, $a = b\epsilon_1$, $b = a\epsilon_2$, then $\epsilon_1\epsilon_2 = 1$, where ϵ_1 and ϵ_2 are integral elements; such integers which are divisors of 1 are called *units in* $\mathfrak{g}$ and elements a and b related as above, *associated elements.*

If a and b are integral, then an element d in $\mathfrak{g}$ is said to be a *greatest common divisor of a and b* if a and b are divisible by d and if d is divisible by every common divisor of a and b. If d is a unit, then a and b are said to be *relatively prime.* $\mathfrak{R}$ is *complete*[3] (relative to $\mathfrak{g}$) if every pair of elements in $\mathfrak{g}$ has a g. c. d.

A *prime element* p in $\mathfrak{g}$ is an integral element that is not a unit and whose divisors are associated with 1 or p. $\mathfrak{R}$ is *primary* (relative to $\mathfrak{g}$) if for every two integers a and b it is true that either a and b are relatively prime or that there exists a common prime element divisor p of a and b. Hence, if $\mathfrak{R}$ is primary, every integer $a \neq 0$ is either a unit or is divisible by a prime element.

The following theorem is proved in a manner very similar to that of a theorem of Prüfer:[4]

THEOREM 1. *If $\mathfrak{R}$ is complete relative to $\mathfrak{g}$, and if $a = a_1 \cdots a_n$ (where a, a_i $(i = 1, \cdots, n)$ are integers) is divisible by b, then $b = b_1 \cdots b_n$, where b_i $(i = 1, \cdots, n)$ is an integer which divides a_i.*

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[1] *Untersuchungen über Teilbarkeitseigenschaften in Körpern,* Journal für Mathematik, vol. 168(1932), pp. 1–36.

[2] That is, $\mathfrak{R}$ is a domain of integrity (Integritätsbereich) with unit element.

[3] Prüfer, op. cit., p. 3.

[4] Loc. cit., Theorem 3.