PROPERTIES OF INVARIANT SETS UNDER POINTWISE PERIODIC HOMEOMORPHISMS

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A single-valued continuous transformation $T(M) = M$ of a compact metric space onto itself is said to be *pointwise periodic* provided that for each point x in M there exists a positive integer n such that $T^n(x) = x$. It follows directly from this definition that such a transformation is one-to-one and hence is a homeomorphism of M onto itself. W. L. Ayres has recently studied this type of homeomorphism together with other related types on locally connected continua.[1] His generosity in discussing these results has interested the authors in this type of transformation and thus led to the present paper.

In contrast with previous papers we require only that the space be compact and metric.

1. **Definitions and preliminary lemmas.** Let M be a compact metric space and $T(M) = M$ a pointwise periodic homeomorphism. Let L (or L_0) be any closed invariant subset of M, i.e., any closed subset of M such that $T(L) = L$. Denote by $p(T, x)$ the period of any point x in M under T, i.e., the least positive integer n such that $T^n(x) = x$. Let L_1 consist of all those points x of L such that for any positive integer N and any neighborhood U of x there exists a point y (distinct from x) in U for which $p(T, y) > N$. In other words, L_1 consists of all points of L at which $p(T, x)$ has an unbounded limit superior. If L_β has been defined for all ordinals β less than a given ordinal α, we may define L_α as follows. In case α is an isolated number, let L_α consist of all those points x of $L_{\alpha-1}$ at which $p(T, x)$ has an unbounded limit superior (T is considered as being defined only on $L_{\alpha-1}$). If α is a limit number, define $L_\alpha = \prod_{\beta<\alpha} L_\beta$. Thus we have defined for every ordinal α a set L_α.

Lemma 1. *For every α the set L_α is closed and invariant.*

Proof. The proof will be by transfinite induction. Assume the lemma true for all ordinals $\beta < \alpha$. Then if α is a limit number, L_α is closed, being the product of closed sets. It is also invariant, since if $p \in L_\alpha$ then $p \in L_\beta$ for all $\beta < \alpha$. Thus $T(p) \in L_\beta$ for all $\beta < \alpha$ since all these sets are invariant by hypothesis. Consequently, $T(p) \in L_\alpha$ so that this set is invariant.

On the other hand, if α is an isolated number, then $L_{\alpha-1}$ exists. For any point $p \in L_\alpha$ we may find a sequence of points $\{p_i\}$ in $L_{\alpha-1}$ such that $\lim p_i = p$

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[1] See Bulletin of the American Mathematical Society, vol. 44(1938), p. 329, abstract no. 172; Bulletin of the American Mathematical Society, vol. 43(1937), p. 20, abstract no. 3; and a forthcoming article in Fundamenta Mathematicae.