# TERNARY TRILINEAR FORMS IN THE FIELD OF COMPLEX NUMBERS

By R. M. Thrall and J. H. Chanler

1. **Introduction**. The specific problem of this paper is a classification of ternary trilinear forms with coefficients in the field of complex numbers. The complete solution is given by the table on page 689, and Theorem 12 gives necessary and sufficient conditions for the equivalence of two ternary trilinear forms under non-singular linear transformations on their sets of variables.

However, the authors consider the introduction of geometric methods to the study of trilinear forms of more importance than the specific results obtained. A trilinear form defines certain algebraic transformations between subspaces (manifolds) of the spaces of the variables. These subspaces may in general be curves, surfaces, or even isolated points; but for ternary trilinear forms they are plane cubic curves and the transformations between them are birational.

Consider the general trilinear form

$$F(x, y, z) = \sum a_{hij} x_h y_i z_j ,$$

where there are $r_h$ $x$'s, $r_i$ $y$'s, $r_j$ $z$'s and the $a_{hij}$ are arbitrary complex numbers. Associated with the form is the three-way matrix $(a_{hij})$, called the matrix of the form. We suppose that the numbers $r_h$, $r_i$, $r_j$ are the smallest numbers of $x$'s, $y$'s, $z$'s, respectively, in terms of which the form can be expressed.[1] Two forms $F$ and $F'$ are called *equivalent*, and we write $F \sim F'$, if $F$ can be sent into $F'$ by non-singular linear transformations on the sets of variables taken separately. The totality of forms equivalent to a given form $F$ is said to constitute a class of forms denoted by $[F]$. If $F \sim F'$, then $[F] = [F']$.

For a given three-way matrix $(a_{hij})$ there are six ways in which sets of variables $x$, $y$, $z$ can be associated with the elements $a_{hij}$ to produce trilinear forms.[2] Two trilinear forms derived from the same matrix are

$$F(x, y, z) = a_{111}x_1y_1z_1 + a_{112}x_1y_1z_2 + a_{113}x_1y_1z_3 + a_{121}x_1y_2z_1 + a_{122}x_1y_2z_2 + a_{123}x_1y_2z_3,$$

with $r_h = 1$, $r_i = 2$, $r_j = 3$, and

$$F(x, z, y) = F'(x, y, z)$$
$$= a_{111}x_1z_1y_1 + a_{112}x_1z_1y_2 + a_{113}x_1z_1y_3 + a_{121}x_1z_2y_1 + a_{122}x_1z_2y_2 + a_{123}x_1z_2y_3,$$

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[1] These numbers are equal to the $h$, $i$, $j$ index ranks introduced by R. Oldenburger, IV. That these index ranks are the smallest numbers of variables in terms of which the form can be expressed was noted by H. R. Brahana, III, pp. 190–191. (Roman numerals refer to the bibliography at the end of paper.)

[2] VI, p. 384.