LINEAR FUNCTIONALS AND COMPLETELY ADDITIVE SET FUNCTIONS

BY B. J. PETTIS

Introduction. We should like first to recall a few well known definitions. Let $T = [t]$ be an abstract space composed of arbitrary elements t, and let $\mathcal{F}^T$ denote the collection of all subsets of T. If $\mathcal{F} = [F]$ is a non-vacuous subcollection of $\mathcal{F}^T$, then $\mathcal{F}$ is an *additive family*[1] if

(I) F in $\mathcal{F}$ implies T $-$ F in $\mathcal{F}$, and

(II) F_n in $\mathcal{F}$ for $n = 1, 2, \cdots$ implies $\sum_n F_n$ in $\mathcal{F}$.

A finite or denumerable aggregate δ of elements $F_1, \cdots, F_n, \cdots$ of $\mathcal{F}$ that are disjoint in pairs will be called a *split*; if δ has only a finite number of F_n's, it is a *finite split*. If Δ is the collection of all finite splits and Δ' that of all splits, then $\alpha(F)$ defined from $\mathcal{F}$ to the reals is *additive* if $\sum_\delta \alpha(F_i) = \alpha(\sum_\delta F_i)$ for every δ in Δ, and *completely additive* (c.a.) if this holds for every δ in Δ'. The *norm* of an additive $\alpha(F)$ is defined to be

$$\| \alpha \| = \text{Var}(\alpha, T) = \underset{\Delta}{\text{l.u.b.}} \sum_\delta | \alpha(F_i) |. \tag{0.1}$$

For such an α it is clear that

$$\underset{\mathcal{F}}{\text{l.u.b.}} | \alpha(F) | \leqq \| \alpha \| \leqq 2(\underset{\mathcal{F}}{\text{l.u.b.}} | \alpha(F) |), \tag{0.2}$$

and hence an additive α is bounded if and only if $\| \alpha \| < \infty$. If α is c.a., then $\| \alpha \| < \infty$ and α must be bounded.[2]

The space $A = [\alpha]$, consisting of the functions $\alpha(F)$ bounded and additive (b.a.) over $\mathcal{F}$, is, under the definition of norm given in (0.1), a linear normed complete space, i.e., a B-space;[3] this is likewise true of the subset $C = [\gamma]$ composed of the functions $\gamma(F)$ completely additive over $\mathcal{F}$. Thus these in particular are B-spaces: the collection A^T of functions b.a. over $\mathcal{F}^T$ and its subset C^T consisting of functions c.a. over $\mathcal{F}^T$.

The following notational rules will be generally obeyed, although not always:

(i) Roman capitals are subsets of T and script capitals are collections of such subsets.

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[1] S. Saks, *Theory of the Integral*, Warsaw, 1937, p. 7. Hereafter we shall refer to this treatise as TI.

[2] TI, p. 10.

[3] S. Banach, *Théorie des Opérations Linéaires*, Warsaw, 1932, p. 53. The letters TOL will refer to this monograph.