THEOREMS ON RIESZ MEANS

By H. L. Garabedian and W. C. Randels

1. **Introduction.** We are concerned in this paper with means of the type

$$\sigma_n = \frac{p_0 s_0 + p_1 s_1 + \cdots + p_n s_n}{P_n}, \tag{1.1}$$

where

$$P_n = p_0 + p_1 + \cdots + p_n ,$$

$\{s_n\}$ is a given sequence, and $\sum p_n$ is a divergent series of positive terms. Since means of the type (1.1) were used in the early development of the Riesz typical means, they are called Riesz means and are designated by (R, p_n).[1] It is of particular interest to notice that all of the Riesz means of the type under consideration constitute regular definitions of summation.

2. **Relative inclusion.** Let us first consider a theorem relating to the relative inclusiveness of these methods.

Theorem 1. *A necessary and sufficient condition that $(R, q_n) \subset (R, p_n)$ is*[2] *that*

$$\frac{1}{P_n} \sum_{\nu=0}^{n-1} \left| \frac{p_\nu}{q_\nu} - \frac{p_{\nu+1}}{q_{\nu+1}} \right| Q_\nu + \frac{p_n Q_n}{q_n P_n} < N, \tag{2.1}$$

where N is independent of n.

This is an improvement on a theorem due to Cesàro[3] and Hardy.[4] Cesàro proved that $(R, q_n) \subset (R, p_n)$ provided that

$$\frac{p_n}{q_n} \geqq \frac{p_{n+1}}{q_{n+1}}, \tag{2.2}$$

while Hardy proved that the same result obtains when

$$\begin{cases} \text{(a)} \quad \dfrac{p_n}{q_n} \leqq \dfrac{p_{n+1}}{q_{n+1}}, \\ \text{(b)} \quad \dfrac{p_n Q_n}{q_n P_n} = O(1). \end{cases} \tag{2.3}$$

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[1] G. H. Hardy, Proc. London Math. Soc., (2), vol. 8(1910), pp. 301–320.

[2] We understand $A \subset B$ to mean that every sequence summable by a method A is also summable by a method B to the same limit, or that the method B includes the method A. Further, we interpret $A \approx B$ to mean that the two methods are equivalent, that is, each method includes the other.

[3] Cesàro, Bulletin des Sciences Mathématiques, (2), vol. 13(1889), pp. 51–54.

[4] Hardy, Quarterly Journal, vol. 38(1907), pp. 269–288.