TENSOR PRODUCTS OF ABELIAN GROUPS

By Hassler Whitney

1. **Introduction.** Let G and H be Abelian groups. Their direct sum $G \oplus H$ consists of all pairs (g, h), with $(g, h) + (g', h') = (g + g', h + h')$. If we consider G and H as subgroups of $G \oplus H$, with elements $g = (g, 0)$ and $h = (0, h)$, then we may form $g + h$, and the ordinary laws of *addition* hold. Our object in this paper is to construct a group $G \circ H$ from G and H, in which we can form $g \cdot h$, with the properties of *multiplication*; that is, the distributive laws

$$(1.1) \qquad (g + g')\cdot h = g\cdot h + g'\cdot h, \qquad g\cdot(h + h') = g\cdot h + g\cdot h'$$

hold. Clearly $G \circ H$ must contain elements of the form $\sum g_i \cdot h_i$; it turns out (Theorem 1) that with these elements, *assuming only* (1.1), *we obtain an Abelian group*, which we shall call the *tensor product* of G and H.[1]

The tensor product is known in one important case; namely, in tensor analysis (see §4, (b), and §11), though the definition in the form here given does not seem to have been used. Certain other cases are known (see §4). We refer to the examples there given for further indications of the scope of the theory. A direct product of algebras has been constructed by J. L. Dorroh,[2] by methods closely allied to those of the present paper.

As is to be expected, we see in Part I that when we multiply several groups together, the associative and commutative laws hold; also the distributive laws with respect to direct sums and difference groups. The group of integers plays the rôle of a unit group.[3] The rest of Part I is devoted largely to a study of the relation between groups with operator rings and tensor products; in particular, divisibility properties are considered.

In Part II, a detailed study of tensor products of linear spaces is made; we now assume $rg\cdot h = g\cdot rh$ (r real). With any element α of $G \circ H$ are associated subspaces $G(\alpha)$ of G and $H(\alpha)$ of H; their dimensions equal the minimum number of terms in an expression $\sum g_i \cdot h_i$ for α, and in this expression the g_i and h_i form bases in $G(\alpha)$ and $H(\alpha)$. The elementary operations of tensor algebra are derived, and a direct manner of introducing covariant differentiation is indicated.[4] If the linear spaces are topological, a topology may be introduced into

Received February 23, 1938; presented to the American Mathematical Society, February 26, 1938. See Proceedings of the National Academy of Sciences, vol. 23(1937), p. 290.

[1] This is so even if G and H are not Abelian; see Theorem 11. If G and H are linear or topological, we use a slightly different definition.

[2] J. L. Dorroh, *Concerning the direct product of algebras*, Annals of Mathematics, vol. 36 (1935), pp. 882–885. The author is indebted to the referee for pointing out this paper to him.

[3] In linear spaces, the group of real numbers also is a unit.

[4] Some of these results have been derived independently by H. E. Robbins.