INTEGRATION IN ABSTRACT METRIC SPACES

By S. Saks

1. In a recent note Banach[1] established the following theorem which may be regarded as an interesting extension of the well-known formula of F. Riesz for linear functionals over the space of continuous functions on a finite interval.

Let $\mathbf{H}$ be a compact metric space and Φ a non-negative linear functional defined over the space of continuous real functions on $\mathbf{H}$; i.e.,

(i) $\Phi(f) \geqq 0$ whenever f is a non-negative continuous function on $\mathbf{H}$,

(ii) $\Phi(f + g) = \Phi(f) + \Phi(g)$ for any two continuous functions f and g on $\mathbf{H}$,

(iii) $\lim_n \Phi(f_n) = 0$, if $\{f_n\}$ is a sequence of continuous functions, converging to 0 uniformly on $\mathbf{H}$.

Then there exists a measure μ in the space $\mathbf{H}$, with respect to which

$$\Phi(g) = \int_{\mathbf{H}} g(x)\, d\mu(x) \tag{1.1}$$

for any function g continuous on $\mathbf{H}$.[2]

Banach's original proof of the above theorem is based on the general theory of functional operations and on his theory of integration on abstract spaces. In this note we give another proof which seems more elementary and which is based directly on the Lebesgue theory of integration as extended to abstract spaces by Fréchet.

2. In what follows $\mathbf{H}$ will be a fixed compact metric space and $\rho(a, b)$ the distance between any two points a, b of the space. If $a \in \mathbf{H}$ and $r > 0$, then $S(a, r)$ will denote the open sphere with center a and radius r, i.e., the set of points x such that $\rho(a, x) < r$. The set of points x such that $\rho(a, x) = r$ is the surface of the sphere $S(a, r)$. If A is any set in $\mathbf{H}$, the closure of A will be denoted, as usual, by $\bar{A}$.

Finally, Φ will denote a non-negative linear functional defined over the space of continuous functions on $\mathbf{H}$.

3. For every set E in $\mathbf{H}$, we shall denote by $\Gamma(E)$ the lower bound of the numbers $\Phi(f)$, where f is an arbitrary non-negative continuous function on $\mathbf{H}$

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[1] S. Banach, *The Lebesgue integral in abstract spaces* (Note II in the book by S. Saks, *Theory of the Integral*, 2d ed., Monografie Matematyczne, Warsawa, 1937, pp. 320–330, esp. p. 326).

[2] In this connection see also G. Fichtenholz and L. Kantorovich, *Sur les opérations dans l'espace des fonctions bornées*, Studia Mathematica, vol. 5(1934), pp. 67–78.