CERTAIN INTEGRALS AND INFINITE SERIES INVOLVING ULTRASPHERICAL POLYNOMIALS AND BESSEL FUNCTIONS

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1. **Introduction.** Let $P_n^{(\lambda)}(x)$ denote the ultraspherical polynomials defined by the generating function $(1 - 2xw + w^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x)w^n$ [cf. **6**, p. 50; **7**, p. 329; **4**, p. 37].[1] The following considerations are devoted to the discussion, first, of the integral formula

$$D_1(\lambda; l, m, n) \equiv \int_{-1}^{1} (1 - x^2)^{\lambda-\frac{1}{2}} P_l^{(\lambda)}(x)P_m^{(\lambda)}(x)P_n^{(\lambda)}(x)\,dx$$

$$(1.1)\qquad = \frac{2^{1-2\lambda}}{\{\Gamma(\lambda)\}^2} \frac{\pi}{s + \lambda} \frac{\Gamma(s + 2\lambda)}{\Gamma(s + 1)} \frac{\binom{s - l + \lambda - 1}{s - l}\binom{s - m + \lambda - 1}{s - m}\binom{s - n + \lambda - 1}{s - n}}{\binom{s + \lambda - 1}{s}} \quad \text{or} \quad 0,$$

second, of the infinite expansion

$$D_2(\lambda; \alpha, \beta, \gamma) \equiv \sum_{n=0}^{\infty} (n + \lambda) \left\{\frac{\Gamma(n + 1)}{\Gamma(n + 2\lambda)}\right\}^2 P_n^{(\lambda)}(\cos \alpha)P_n^{(\lambda)}(\cos \beta)P_n^{(\lambda)}(\cos \gamma)$$

$$(1.2)\qquad = 2^{-2\lambda}\pi\{\Gamma(\lambda)\}^{-4}\{\sin \alpha \sin \beta \sin \gamma\}^{1-2\lambda} \left\{\sin \frac{\alpha + \beta + \gamma}{2} \sin \frac{\beta + \gamma - \alpha}{2} \sin \frac{\gamma + \alpha - \beta}{2} \sin \frac{\alpha + \beta - \gamma}{2}\right\}^{\lambda-1} \quad \text{or} \quad 0,$$

third, of the integral formula

$$(1.3)\qquad S(\nu; a, b, c) \equiv \int_0^{\infty} J_\nu(ax)J_\nu(bx)J_\nu(cx)x^{1-\nu}\,dx = \frac{2^{\nu-1}\Delta^{2\nu-1}}{\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2})(abc)^\nu} \quad \text{or} \quad 0.$$

In (1.1) we have $\lambda > -\frac{1}{2}$ and the numbers l, m, n are arbitrary non-negative integers. The first of the two given values holds if $l + m + n$ is even, $l + m + n = 2s$, and a triangle exists with the sides l, m, n; and the second value holds in every other case.[2]

In (1.2) we have $\lambda > 0$ and the parameters α, β, γ are arbitrary positive

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[1] The bold face numbers refer to the bibliography at the end.

[2] In case $\lambda = 0$ the formula needs a slight modification.