A COTANGENT ANALOGUE OF CONTINUED FRACTIONS

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The continued iteration of a rational function $f(x, y)$ of two variables provides an algorithm for the expression of a real number as a sequence of rational numbers. Thus the function

$$f(x_1, f(x_2, f(x_3, \cdots))) \tag{1}$$

becomes an infinite series for $f(x, y) = x + y$ and an infinite product for $f(x, y) = xy$. For $f(x, y) = x + 1/y$ we obtain the regular continued fraction

$$x_1 + \cfrac{1}{x_2 + \cfrac{1}{x_3 + \cdots}} = x_1 + \frac{1|}{|x_2} + \frac{1|}{|x_3} + \cdots.$$

By far the most frequently used function is $f(x, y) = x + y/c$, which gives the "power series"

$$x_1 + \cfrac{x_2 + \cfrac{x_3 + \cdots}{c}}{c} = x_1 + \frac{x_2}{c} + \frac{x_3}{c^2} + \cdots,$$

where the x's are the coefficients, used when $c = 10$ for the decimal representation of real numbers.[1] The algorithm associated with $f(x, y) = x(1 - y)$ has been discussed by T. A. Pierce.[2]

This paper is concerned with the case of

$$f(x, y) = (xy + 1)/(y - x) = \cot(\operatorname{arc\,cot} x - \operatorname{arc\,cot} y),$$

so that (1) becomes the function

$$\cot(\operatorname{arc\,cot} x_1 - \operatorname{arc\,cot} x_2 + \operatorname{arc\,cot} x_3 - \cdots).$$

This function, despite its aspect, is no more transcendental than a regular continued fraction and both functions have many properties in common. Furthermore, in order to obtain sequences of rational approximations to a real number, we specialize the x's to be integers, as in the continued fraction, and consider therefore expressions of the form

$$\cot \sum_{\nu=0} (-1)^{\nu} \operatorname{arc\,cot} n_{\nu}, \tag{2}$$

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[1] This use of the function $x + y/c$ is at least 4000 years old. See Amer. Jour. of Semitic Languages and Literature, vol. 36(1920), No. 4. The Babylonians used $c = 60$.

[2] Amer. Math. Monthly, vol. 36(1929), pp. 523–525.