THE ULTRAHYPERBOLIC DIFFERENTIAL EQUATION WITH FOUR INDEPENDENT VARIABLES

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The properties of a linear homogeneous partial differential equation of the second order

$$(1) \qquad \sum_{i,k} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_i b_i \frac{\partial u}{\partial x_i} + cu = 0$$

for a function $u(x_1, \cdots, x_n)$ are known to depend largely on the index of the quadratic form $Q(\xi) = \sum_{i,k} a_{ik} \xi_i \xi_k$.[1] If by a suitable real linear transformation Q can be brought into the form $\pm(\xi_1^2 + \cdots + \xi_n^2)$, i.e., if Q is definite, (1) is called an *elliptic* equation. If Q can be transformed into $\pm(\xi_1^2 + \cdots + \xi_{n-1}^2 - \xi_n^2)$, the equation is called *normal hyperbolic*. Elliptic and normal hyperbolic equations constitute the two types which have been studied more extensively, besides the case of a parabolic equation for which $\det(a_{ik}) = 0$. Equations which are neither elliptic, nor parabolic, nor normal hyperbolic, i.e., equations for which the corresponding quadratic form Q can be written in the form $\xi_1^2 + \xi_2^2 \pm \cdots \pm \xi_{n-2}^2 - \xi_{n-1}^2 - \xi_n^2$, have scarcely been treated, at least not without restriction to solutions which are analytic in all or some of the variables. For such equations the notation *ultrahyperbolic* has been introduced by R. Courant.

Ultrahyperbolic equations occur only if the number of independent variables is at least 4. The simplest example is the equation

$$(2) \qquad \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = 0,$$

which forms the subject of the present paper. Obviously every ultrahyperbolic equation $\sum_{i,k} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} = 0$ with constant coefficients can be transformed into (2).

The theory of ultrahyperbolic equations with constant coefficients has been made accessible recently by the discovery by L. Asgeirsson of a functional equation for the solutions of any second order differential equation with constant coefficients of any type whatsoever.[2] For our equation (2) this functional equation takes the form

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[1] Cf. Hadamard, *Lectures on Cauchy's Problem*, Book I, Chapter II.

[2] Cf. Asgeirsson, *Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten*, Mathematische Annalen, vol. 113(1936), pp. 321–346. An exposition of Asgeirsson's results and further applications can be found in the second volume of Courant-Hilbert, *Methoden der Mathematischen Physik*. Cf. also H. Poritsky, *Generalizations of the Gauss law of the spherical mean*, Transactions of the American Mathematical Society, vol. 43(1938), p. 215.