SOME EXISTENCE THEOREMS FOR PROBLEMS IN THE CALCULUS OF VARIATIONS

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Introduction. In a recent paper[1] I have established a theorem on semi-continuity of integrals of the calculus of variations under hypotheses weak enough to apply to the parametric and ordinary problems, as well as to several other problems. Here I wish to establish existence theorems of a comparable generality for the parametric and ordinary problems, as well as for problems involving higher derivatives.

The added generality in the parametric problem is not very important. It consists merely of a relaxing of the continuity requirements on the integrand; instead of being required to have certain partial derivatives, the integrand is required only to be a lower semi-continuous function of its arguments. The chief point of interest is that the existence theorem for the parametric problem is obtained without added effort as a special case of one of the auxiliary theorems designed to handle the problem in ordinary form.

With problems in ordinary form the situation is quite different. The existence theorems for such problems may be roughly classified into two types: those which depend on the behavior of the integrand $f(x, y, y')$ as $| y' | \rightarrow \infty$, and those which depend on the differential properties of minimizing curves. The second type is not considered in this paper. In the first type, a fundamental theorem is the one[2] which applies to quasi-regular integrands for which

$$(*) \qquad f(x, y, y')/| y' | \rightarrow \infty \text{ as } | y' | \rightarrow \infty .$$

This requirement implies in particular that $\int f dx$ is "positive quasi-regular semi-normal", in Tonelli's terminology. Clearly the requirement that (*) hold everywhere can be relaxed in two ways. We may suppose that (*) fails to hold on a set E, but $\int f dx$ remains positive quasi-regular semi-normal on E. The question is, how general can the set E be without disturbing the existence of the solution? In this paper a class of sets (progressively distributed sets) is defined, and it is shown that E may be any progressively distributed set. All previously known classes of sets E are contained in this class. A different way of relaxing

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[1] *Semi-continuity of integrals in the calculus of variations*, this Journal, vol. 2 (1936), p. 597. This paper will henceforth be referred to as SC.

[2] M. Nagumo, *Über die gleichmässige Summierbarkeit und ihre Anwendung auf ein Variationsproblem*, Japanese Journal of Mathematics, vol. 6 (1929), pp. 173–182.

E. J. McShane, *Existence theorems for ordinary problems of the calculus of variations*, Annali della R. Sc. Norm. Sup. di Pisa, ser. II, vol. 3 (1934), p. 298.

L. Tonelli, *Su gli integrali del calcolo delle variazioni in forma ordinaria*, Annali della R. Sc. Norm. Sup. di Pisa, ser. II, vol. 3 (1934), p. 400.