THE HALF-GROUP OF COSETS BELONGING TO A GROUP

By Charles Hopkins

The problem of incorporating into a single system the various quotient-groups associated with a given group G has recently received attention: one finds, for example, a solution in the papers of Ore on structures.[1] As Ore points out, however, the wide applicability of his results is attained "by the elimination of the elements from the algebraic theories". It is the purpose of this paper to present a solution in which the elements of the quotient-groups occupy the center of interest. We shall incorporate the elements of certain quotient-groups associated with G into a multiplicative system, which we call *the half-group belonging to G.* It is not difficult to see that this "multiplicative system" can never be a group if, as seems reasonable, we require that two elements belonging to distinct quotient-groups have a unique product.

I. The half-group $\Gamma(G)$

Let G denote any group containing more than one element, and let Φ denote a set of operators for G, each operator effecting a *proper* automorphism of G. Of the set Φ we require that it contain operators effecting each of the inner isomorphisms[2] of G. Let $H(G)$ denote the set of all subgroups in G which individually admit each operator of Φ. Evidently $H(G)$ is either the set of all normal subgroups of G or a subset of this set. In any case, $H(G)$ will contain both the identity subgroup E and the group G itself. The following are familiar results: if H_a and H_b are any two members of $H(G)$, then the complexes H_aH_b and H_bH_a are identical and each is equal to the union $\{H_a, H_b\}$; both the union and the cross-cut $[H_a, H_b]$ are contained in $H(G)$.

Now each H in $H(G)$ gives rise to the quotient-group $\Gamma = G/H$. Let $Q(G)$ denote the set of *distinct* quotient-groups associated with the set $H(G)$, two quotient-groups Γ_a and Γ_b being regarded as distinct if, and only if, $H_a \neq H_b$. We suppose, furthermore, that two distinct quotient-groups have no element in common. Let Σ denote the set of all group-elements in $Q(G)$; i.e., the logical sum of the sets of elements in Γ_a, Γ_b, etc. We wish to define for the elements of Σ a "multiplication" which shall have the following characteristics:

(1a) the set Σ is closed under multiplication;

(1b) multiplication is associative for any three elements of Σ;

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[1] *On the foundations of abstract algebra*, I, II, Annals of Mathematics, vol. 36 (1935), pp. 406–437; vol. 37 (1936), pp. 265–292; *Structures and group theory*. I, this Journal, vol. 3 (1937), pp. 149–174.

[2] If G is abelian, the set Φ may be void. Throughout this article we designate simple and multiple "isomorphisms" by the terms *isomorphism* and *homomorphism*, respectively.