THE MAPPING OF BETTI GROUPS UNDER INTERIOR TRANSFORMATIONS

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1. In this paper results will be established from which it follows that the one-dimensional rational Betti group of a compact metric set A under any interior transformation[1] on A maps homomorphically onto the corresponding group of the image of A provided the one-dimensional Betti number $p^1(A)$ is finite. Thus in any case the one-dimensional Betti number of a compact set is not increased[2] when the set undergoes an interior transformation. This presents, in part, a solution to a problem proposed by Eilenberg.[3] However, by making use of what are termed *nodal subsets* of a compact set, we are able to obtain results of considerably increased generality and precision.

We begin with an elementary characterization of interior transformations.

2. Theorem. *Let A and B be compact and $T(A) = B$ be continuous. In order that T be an interior transformation, it is necessary and sufficient that for any $\epsilon > 0$ there exist a $\delta > 0$ such that for any $y \in B$ and any $x \in T^{-1}(y)$, $T[V_\epsilon(x)] \supset V_\delta(y)$.*[4]

Proof. To prove the sufficiency, let U be any open set in A, let $T(U) = V$. Let $y \in V$, $x \in U \cdot T^{-1}(y)$ and let $\epsilon > 0$ be chosen so that $V_\epsilon(x) \subset U$. Then by hypothesis $T[V_\epsilon(x)] \supset V_\delta(y)$ for some $\delta > 0$. This gives

$$V = T(U) \supset T[V_\epsilon(x)] \supset V_\delta(y)$$

and hence V is open in B.

To establish the necessity of the condition, suppose on the contrary that for some $\epsilon > 0$ there exist a sequence y_i in B, $y_i \rightarrow y \in B$, a sequence x_i in A, $x_i \rightarrow x \in T^{-1}(y)$, $x_i \in T^{-1}(y_i)$, and a sequence of numbers $0 < \delta_i \rightarrow 0$ such that for no i does $T[V_\epsilon(x_i)] \supset V_{\delta_i}(y_i)$. But now $T[V_{\frac{1}{2}\epsilon}(x)] = V$ is open and contains

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[1] A continuous transformation $T(A) = B$ is *interior* provided every open set in A maps into an open set in B. See Stoïlow, Annales Scientifiques de l'Ecole Normale Supérieure, vol. 63 (1928), pp. 347–382; we follow the usual custom of omitting Stoïlow's second condition that no continuum in A map into a single point in B.

[2] For the case of this result where A is a graph or a 1-dimensional locally connected continuum, see my paper *Interior transformations on compact sets*, this Journal, vol. 3 (1937), pp. 370-381.

[3] Fundamenta Mathematicae, vol. 24 (1935), p. 175.

[4] We employ the notation $V_r(X)$ for the ϵ-neighborhood of the set X, i.e., the set of all points y such that $\rho(x, y) < r$ for some $x \in X$.