# TRIGONOMETRIC APPROXIMATION IN THE MEAN

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The following theorem is stated without proof by G. H. Hardy and J. E. Littlewood.[1]

THEOREM. *The class* Lip$(\alpha, p)$ *is identical with the class of functions* $f(x)$ *approximable in the mean p-th power, with error* $O(n^{-\alpha})$, *by trigonometrical polynomials of degree n.*

They remark in addition: *This approximation may be made in general by the Fourier polynomials of* $f(x)$; *the case* $p = \infty$, *in which this is not true, is exceptional.*

The initial purpose of this paper is to examine the range of values of $p$ and $\alpha$ for which this theorem and remark are true and to supply proofs. In doing this, related theorems are obtained in which the approximations are in terms of the metric of a more extensive space than $L_p$ and in which the functions that measure the degree of approximation are more general than $n^{-\alpha}$. These theorems and their proofs parallel to a large extent the theorems given by de la Vallée Poussin[2] and Dunham Jackson[3] for the class Lip$(\alpha)$.

We assume throughout that our functions $f(x)$ are periodic with the period $2\pi$. The functions $\Phi(u)$ and $\Psi(u)$ are of Young's type.[4] That is, $\Phi(u)$ is non-negative, convex, and satisfies the relations $\Phi(0) = 0$ and $\Phi(u)/u \to \infty$ as $u \to \infty$; $\Psi(u)$ has similar properties and is such that Young's inequality

$$uv \leqq \Phi(u) + \Psi(u), \qquad u, v \geqq 0$$

holds. Throughout the paper we will write $\Phi \mid u \mid$, $\Psi \mid u \mid$, for $\Phi(\mid u \mid)$, $\Psi(\mid u \mid)$.

If $f(x)$ is measurable and such that $\int_0^{2\pi} \Phi \mid f \mid dx$ exists, $f(x)$ is said to belong to the space $L_\Phi(0, 2\pi)$. If $f(x)$ is such that the product $f(x)g(x)$ is integrable for every $g(x) \epsilon L_\Psi$, then $f(x) \epsilon L_\Phi^*$. For this space

$$\| f \|_\Phi = \sup_g \left| \int_0^{2\pi} f(x)g(x)\, dx \right|$$

for all measurable $g(x)$ with $\rho_g \equiv \int_0^{2\pi} \Psi \mid g \mid dx \leqq 1$. This space[5] is linear,

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[1] *A convergence criterion for Fourier series*, Math. Zeit., vol. 28 (1928), pp. 612–634; in particular, p. 633.

[2] *Leçons sur l'Approximation des Fonctions*, Paris, 1919. We shall refer to this treatise as (P).

[3] *The Theory of Approximation*, New York, 1930. We shall refer to this treatise as (D).

[4] A. Zygmund, *Trigonometrical Series*, Warsaw, 1935, §§4.11, 4.142. We shall refer to this treatise as (Z). It contains extensive bibliographical references to original sources.

[5] (Z), §4.541.