ANALYTIC MAPPING OF COMPACT RIEMANN SPACES INTO EUCLIDEAN SPACE

By S. Bochner

Recently H. Whitney[1] has proved that an n-dimensional separable coördinate space S of class C_q ($1 \leqq q \leqq \infty$) can be mapped topologically onto the Euclidean E_{2n+1} in such a manner that the mapping functions

$$t_\nu = t_\nu(x_1, \cdots, x_n) \qquad (\nu = 1, \cdots, 2n+1) \tag{1}$$

belong to class C_q on S and have a Jacobian of rank n throughout S; the quantities $x_1, \cdots, x_n$ in (1) are local coördinates on S varying with the neighborhood.

It is not known whether for an analytic space S the mapping functions (1) can be chosen analytic. It is the purpose of the present paper to point out that they can be so chosen provided S is compact and has an analytic Riemann metric.

The line of reasoning is very simple. With the fundamental tensor $g_{ij}(x)$, we form the Laplacian

$$\Delta\varphi = \frac{1}{g^{\frac{1}{2}}}\frac{\partial}{\partial x_i}\left(g^{\frac{1}{2}} g^{ij} \frac{\partial \varphi}{\partial x_j}\right) = g^{ij}\varphi_{,ij}\,. \tag{2}$$

In the Hilbert space H of all square integrable functions on S, the Laplacian is essentially the inverse of a completely continuous operator. Therefore the solutions φ of the equation

$$\Delta\varphi = \lambda\varphi \tag{3}$$

form a complete basis in H. But the solutions of (3) are analytic if the coefficients g_{ij} are so. Hence every function $t(x)$ on S is the limit in square mean of analytic functions. For differentiable functions on S we shall prove more; if $t(x)$ belongs to C_∞, then corresponding to any $\epsilon > 0$ there exists an analytic function $\psi^\epsilon(x)$ such that *the function* $t(x) - \psi^\epsilon(x)$ *and its gradient* differ by less than ϵ throughout S.[2] We now apply this approximation to the functions (1). It is easy to see that, for ϵ sufficiently small, the approximating transformation

$$t_\nu = \psi_\nu^\epsilon(x_1, \cdots, x_n) \tag{4}$$

will have the same mapping properties as the original transformation (1), which proves our assertion.

Received February 16, 1937.

[1] *Differentiable Manifolds*, Annals of Math., vol. 37 (1936), pp. 645–680; p. 654, Theorem 1.

[2] In order to prove this conclusion it would be sufficient to assume that $t(x)$ belongs to C_2. But the proof would become more elaborate.