ORTHOGONAL POLYNOMIALS ON A PLANE CURVE

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1. **Introduction.** Polynomials in two real variables x and y orthogonal with respect to integration along a curve in the (x, y)-plane can be constructed by the usual process for building up a system of orthogonal functions. If the curve is not algebraic, they have formal properties closely corresponding to those of polynomials orthogonal over a two-dimensional region.[1] For an algebraic curve the relations are different in important respects, reverting in some degree toward those which are familiar in the case of orthogonal functions of a single variable. This is to be pointed out in detail below, under hypotheses which, though not of the utmost generality, are still sufficiently illustrative.

2. **Orthogonal polynomials on a non-algebraic curve.** Let $\varphi(t)$, $\psi(t)$ be continuous functions of t, of period A. If they are not both constant, the equations $x = \varphi(t)$, $y = \psi(t)$ may be regarded as defining a closed curve C (not necessarily of simple character). A relation of linear dependence connecting any finite number of the functions $[\varphi(t)]^h$, $[\psi(t)]^k$, $h = 0, 1, 2, \cdots, k = 0, 1, 2, \cdots$, would mean that a polynomial in x and y vanishes identically on the curve, and so that the curve is the locus or a part of the locus of an algebraic equation. *Let it be assumed for the present that no such relation of linear dependence exists.*

Let $\rho(t)$ be a non-negative integrable function of period A, which, if not everywhere positive, is at any rate such that its product with any polynomial in $\varphi(t)$ and $\psi(t)$ (having a non-vanishing coefficient) is different from zero for a set of values of t of positive measure in a period. This condition will be satisfied, for example, if there is an interval in which $\rho(t)$ is almost everywhere different from zero and for which the corresponding points (x, y) do not belong to an algebraic locus.

Under the hypotheses that have been formulated any finite number of the quantities $\rho^{\frac{1}{2}}$, $\rho^{\frac{1}{2}}x$, $\rho^{\frac{1}{2}}y$, $\rho^{\frac{1}{2}}x^2$, $\rho^{\frac{1}{2}}xy$, $\rho^{\frac{1}{2}}y^2$, $\cdots$, regarded as functions of t for $0 \leqq t \leqq A$, are linearly independent. It is possible by "Schmidt's process of orthogonalization" to construct from them a sequence of functions which are orthogonal and normalized over the interval. When the functions are taken in the order indicated, the members of the orthogonal set are of the form $[\rho(t)]^{\frac{1}{2}}q_{nm}(x, y)$, $n = 0, 1, 2, \cdots, m = 0, 1, \cdots, n$, where $x = \varphi(t)$, $y = \psi(t)$, and q_{nm} is a polynomial of degree n in x and y together, while m is the exponent

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[1] Cf. D. Jackson, *Formal properties of orthogonal polynomials in two variables*, this Journal, vol. 2 (1936), pp. 423–434; referred to hereafter as paper A.