SUMMABILITY OF CONJUGATE DERIVED SERIES

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We consider a function $f(x)$ with period 2π and integrable over $(-\pi, \pi)$. The Fourier series of such a function is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

The conjugate derived series of $f(x)$ is

$$\sum_{n=1}^{\infty} n(a_n \cos nx + b_n \sin nx).$$

We define

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x)$$

and

$$\Phi(t) = \frac{-1}{4\pi}\int_t^{\pi} \varphi(y) \csc^2 \tfrac{1}{2}y\, dy.$$

Throughout this paper we shall suppose that $\varphi(t)/t \subset L$ on $(-\pi, \pi)$. This implies that $\Phi(t) \subset L$, since

$$\int_0^{\pi} |\Phi(t)|\, dt \leqq \frac{1}{4\pi}\int_0^{\pi} dt \int_t^{\pi} |\varphi(y) \csc^2 \tfrac{1}{2}y|\, dy = \frac{1}{4\pi}\int_0^{\pi} dy\, |\varphi(y)\csc^2 \tfrac{1}{2}y| \int_0^{y} dt$$
$$= O\left(\int_0^{\pi} |\varphi(y)| \frac{dy}{y}\right).$$

If $\alpha > 1$ and

$$\frac{\alpha - 1}{t^{\alpha-1}}\int_0^t \Phi(y)(t-y)^{\alpha-2}\, dy \to S \quad \text{as} \quad t \to +0,$$

we say that

$$\text{conj. der. lim } \varphi(t) = S(R', \alpha).$$

If $0 < \alpha \leqq 1$, and for some $\beta > 1$

$$\frac{\beta - 1}{t^{\beta-1}}\int_0^t \Phi(y)(t-y)^{\beta-2}\, dy \to S \quad \text{as} \quad t \to +0,$$

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