STRUCTURES AND GROUP THEORY. I

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In a recent paper on the foundations of abstract algebra[1] I have shown that the principal results on algebraic domains are not primarily to be considered as properties of the elements of the domain itself but as properties of certain systems of distinguished subsets, like systems of subgroups, ideals, submoduli, etc. These systems of subsets have the common characteristic property that they form a *structure*, i.e., a system in which *union* and *cross-cut* of two elements are defined. The theorems on algebraic domains are shown to be theorems on structures. This explains the well-known similarity of several algebraic theories and makes possible a unified structural theory applicable to all systems.

After this common foundation for the algebraic theories has been established, it is, however, not difficult to see that the various algebraic domains like fields, rings and groups have peculiar structural properties of their own. In certain cases it is even possible to characterize the domains by these properties.

In the following we shall apply the principles of the theory of structures to the foundation of the theory of groups, that is, we base the theory as far as possible directly upon the properties of subgroups and eliminate the elements from theorems and proofs. This entails a certain simplification. More important, however, is the fact that this method, even in the elementary theory of groups which we consider in this paper, leads to new systematic points of view and interesting new results.

In Chapter I we discuss the structure formed by all subgroups of a given group and indicate the general principle of duality. Furthermore, in the theory of structures we have constructed a quotient structure for any structure, while quotient systems A/B in groups have been defined only when B is normal in A. Hence we are led to the introduction of quotient systems for all subgroups. The algebraic system A/B is then a *multi-group* differing from ordinary groups only in the property that the product is not unique.

In Chapter II we consider the law of isomorphism. When the assumption of isomorphism is weakened to co-set correspondence we are led to permutable groups. Such groups have the structural property expressed in Theorem 6. When structure isomorphism is required, one is led to the important type of subgroups which I have called quasi-normal subgroups.

Some of the principal new results are to be found in Chapter III. The

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[1] Oystein Ore, *On the foundation of abstract algebra* I, Annals of Math., vol. 36 (1935), pp. 406–437; II, ibid., vol. 37 (1936), pp. 265–292. These two papers will be cited as *Foundations* I and II.