# SOLUTION OF A PROBLEM OF F. RIESZ ON THE HARMONIC MAJORANTS OF SUBHARMONIC FUNCTIONS

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**Introduction.** Let $u(x, y)$ be a subharmonic function[1] in a domain $G$. Consider a domain $G'$ comprised in $G$ together with its boundary $B'$. If $H(x, y)$ is continuous in $G' + B'$ and harmonic in $G'$, and if $H \geqq u$ on $B'$, then $H \geqq u$ in $G'$ also, by the definition of a subharmonic function. If $u$ is continuous, and if the Dirichlet problem is solvable for the region $G' + B'$, then the harmonic function $\bar{h}$ determined by the condition $\bar{h} = u$ on $B'$ is clearly the one which yields the best possible limitation for $u$ on the basis of the fundamental property of subharmonic functions quoted above.

If however $u$ is a general (and therefore possibly discontinuous) subharmonic function, then the situation is less clear. It can be shown (R 2, p. 358) that there exists in $G'$ a *least* harmonic majorant $h^*$ characterized by the following properties. (a) $h^* \geqq u$ in $G'$, (b) if $H$ is harmonic in $G'$ and $H \geqq u$ in $G'$, then $H \geqq h^*$ in $G'$. But this *least* harmonic majorant did not seem to be the *best* one, as far as usefulness was concerned. At any rate, F. Riesz (R 1, p. 334) reserved the name of best harmonic majorant for a harmonic majorant defined in a different fashion, namely, in terms of the values of $u$ on the boundary $B'$ of $G'$ (see 1.2), while the least harmonic majorant $h^*$ is defined in terms of the values of $u$ in $G'$ alone. *The best harmonic majorant*, in the sense of F. Riesz, will be denoted by $\bar{h}$ and will be referred to by the letters B. H. M. The letters L. H. M. and the notation $h^*$ will refer to *the least harmonic majorant* described above.

F. Riesz stated (R 1, footnote on p. 334) that he established the identity of $\bar{h}$ and $h^*$ in various special cases. Brelot[2] gave an explicit proof in the case when the subdomain $G'$ is bounded by circles. *It is the purpose of this paper to prove the identity of $\bar{h}$ and $h^*$ without any restrictions on $G'$, except for the assumption, implied in the very definition of $\bar{h}$, that the Dirichlet problem is solvable for the region $G' + B'$.*[3]

The proof of this result could be based on the general theorems of F. Riesz

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[1] See F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, parts I and II, Acta Mathematica, vol. 48 (1926), pp. 330–343 and vol. 54 (1930), pp. 322–360. These papers will be referred to as R 1 and R 2.

[2] M. Brelot, *Etude des fonctions sousharmoniques au voisinage d'un point*, Actualités scientifiques et industrielles, vol. 139 (1934), p. 18.

[3] In §5 of this paper, we shall give an interpretation of this result which seems to express more adequately its true meaning.